East Asian Journal on Applied Mathematics Vol. **8**, No. 1, pp. 70-81
doi: 10.4208/eajam.261016.260817a February 2018

An Accelerated SOR-Like Method for Generalised Saddle Point Problems

Ting-Ting Feng[1], Guo-Liang Chen[1] and Xue-Ping Guo[1,*]

[1] *Department of Mathematics, Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, East China Normal University, Shanghai, China.*

Received 26 October 2016; Accepted (in revised version) 26 August 2017.

Abstract. A recent accelerated SOR-like method for generalised saddle point problems is discussed. Sufficient conditions for convergence are derived, and some numerical experiments illustrate its effectiveness.

AMS subject classifications: 65F08, 65F10, 65F50

Key words: Accelerated SOR-like method, generalised saddle point problem, convergence.

1. Introduction

Generalised saddle point problems arise in constrained quadratic programming, constrained least squares problems, mixed finite-element approximations of elliptic PDEs, computational fluid dynamics, and Stokes problems [1, 6, 9, 10, 22]. Let m, n be integers such that $m \geq n > 0$. We consider the generalised saddle point problem

$$\begin{pmatrix} A & B \\ B^{\mathsf{T}} & -C \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{q} \end{pmatrix}, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times m}$ and $C \in \mathbb{R}^{n\times n}$ are respectively symmetric positive definite and symmetric positive semi-definite matrices, B^{T} is the transpose of a full column rank matrix $B \in \mathbb{R}^{m\times n}$, and $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{q} \in \mathbb{R}^n$ are given vectors. In the special case $C = 0$, the problem (1.1) obviously is reduced to the augmented system of linear equations

$$\begin{pmatrix} A & B \\ B^{\mathsf{T}} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{q} \end{pmatrix}. \tag{1.2}$$

Various iteration methods have been used to solve such problems — e.g. Uzawa-type methods [8, 11, 12, 19, 25], Hermitian and skew-Hermitian splitting (HSS) iteration [2–5], preconditioned Krylov subspace methods [1,21], restrictively preconditioned conjugate

*Corresponding author. *Email addresses:* tofengtingting@163.com (T.-T. Feng), glchen@math.ecnu.edu.cn (G.-L. Chen), xpguo@math.ecnu.edu.cn (X.-P. Guo)

gradient methods [6,23] and the successive overrelaxation (SOR) method [24], while Benzi *et al.* [10] reviewed existing approaches. For the augmented linear system (1.2), Golub *et al.* [16] proposed an SOR-like method, which was then further developed [7,13,14]. For the generalised saddle point problem (1.1), Cao [12] discussed the convergence of a nonlinear Uzawa algorithm; Bai & Wang [8] studied parameterised inexact Uzawa methods (PIU), Zhou & Zhang [25] proposed a generalisation of the parameterised inexact Uzawa methods (GPIU), and Huang & Ma [18] developed a new GSOR method. Refs. [8,18,25] deal with symmetric positive definite matrices C, Miao & Cao [19] discussed the GPIU method [25] for symmetric positive semidefinite matrices C under the conditions $\ker(C)\cap\ker(B^\intercal)=0$ with the rank p of C such that $0<p<n$.

Recently, Njeru & Guo [20] have considered an accelerated SOR-like method (ASOR) for the augmented linear system (1.2). Here we apply the ASOR method to the generalised saddle point problem (1.1) when C is a symmetric positive semidefinite matrix, consider properties of the eigenpairs of the iteration matrix, and establish sufficient convergence conditions for this method. In Section 2, we introduce the ASOR method for the problem (1.2), and discuss convergence for the generalised saddle point problem (1.1) in Section 3. Numerical experiments presented in Section 4 illustrate the efficiency of the method.

2. The ASOR Method

Let us rewrite the augmented linear system (1.2) as

$$\begin{pmatrix} A & B \\ -B^\intercal & 0 \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}=\begin{pmatrix} \mathbf{b} \\ -\mathbf{q} \end{pmatrix}$$

with the coefficient matrix

$$\mathscr{A}=\begin{pmatrix} A & B \\ -B^\intercal & 0 \end{pmatrix}=D-L-U\,,$$

where

$$D=\begin{pmatrix} \alpha A & 0 \\ 0 & Q \end{pmatrix},\quad L=\begin{pmatrix} -A & 0 \\ B^\intercal & \frac{1}{2}Q \end{pmatrix},\quad U=\begin{pmatrix} \alpha A & -B \\ 0 & \frac{1}{2}Q \end{pmatrix},$$

involve a positive number α and a symmetric positive definite matrix $Q\in\mathbb{R}^{n\times n}$.

Let ω be a positive number. In the ASOR method we seek the solution of the problem (1.2) by the iteration scheme

$$\begin{aligned}\mathbf{x}^{(k+1)}&=\frac{\alpha}{\alpha+\omega}\mathbf{x}^{(k)}-\frac{\omega}{\alpha+\omega}A^{-1}(B\mathbf{y}^{(k)}-\mathbf{b})\,,\\ \mathbf{y}^{(k+1)}&=\mathbf{y}^{(k)}+\frac{2\omega}{2-\omega}Q^{-1}(B^\intercal\mathbf{x}^{(k+1)}-\mathbf{q})\,,\end{aligned}$$