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A Convex and Exact Approach to Discrete Constrained TV-L1 Image Approximation

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Abstract. We study the TV-L1 image approximation model from primal and dual perspective, based on a proposed equivalent convex formulations. More specifically, we apply a convex TV-L1 based approach to globally solve the discrete constrained optimization problem of image approximation, where the unknown image function $u(x) \in \{f_1, \dots, f_n\}$, $\forall x \in \Omega$. We show that the TV-L1 formulation does provide an exact convex relaxation model to the non-convex optimization problem considered. This result greatly extends recent studies of Chan et al., from the simplest binary constrained case to the general gray-value constrained case, through the proposed rounding scheme. In addition, we construct a fast multiplier-based algorithm based on the proposed primal-dual model, which properly avoids variability of the concerning TV-L1 energy function. Numerical experiments validate the theoretical results and show that the proposed algorithm is reliable and effective.

Key words: Convex optimization, primal-dual approach, total-variation regularization, image processing.

1. Introduction

Many tasks of image processing can be formulated and solved successfully by convex optimization models – e.g. image denoising [21, 24], image segmentation [5], image labeling [4, 22] etc. The reduced convex formulations can be studied in a mathematically sound way and usually tackled by a tractable numerical scheme. Minimizing the total-variation function for such convex image processing formulations is of great importance [5, 6, 17–20, 24, 27], as it preserves edges and sharp features.

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In their pioneer works [8,9], Chan et al. proposed the TV-L1 regularized image approximation model

$$\min_u \left\{ P(u) := \alpha \int_\Omega \left| f - u \right| dx + \int_\Omega |\nabla u(x)| \, dx \right\}, \tag{1.1}$$

which was first introduced and studied by Alliney [1,2] for discrete one-dimensional signals' denoising. Chan et al. [8,9] demonstrated an interesting property of the TV-L1 model (1.1) – viz. that for the input binary image $f(x) \in \{0,1\}$, there exists at least one global optimum $u(x) \in \{0,1\}$. It follows that the convex TV-L1 formulation (1.1) actually solves the nonconvex optimization problem

$$\min_{u(x) \in \{0,1\}} \alpha \int_\Omega \left| f - u \right| dx + \int_\Omega |\nabla u(x)| \, dx , \tag{1.2}$$

globally and exactly! Hence (1.1) provides an exact convex relaxation of the binary constrained optimization problem (1.2). Chan et al. [8, 9] also proved that rounding the computed result of (1.1) may give a series of global optima of the binary constrained optimization model (1.2).

Previous work and motivation

With the help of co-area formula, Chan et al. [8,9] proved that the energy functional $P(u)$ of (1.1) can be represented in terms of the upper level-set sequence of the image functions $u(x)$ and $f(x)$ – i.e.

$$P(u) = \int_{-\infty}^{+\infty} \left\{ |\partial U^\gamma| + \alpha \left| U^\gamma \triangle F^\gamma \right| \right\} d\gamma , \tag{1.3}$$

where U^γ and F^γ denote the $\gamma-$upper level set of the unknown $u(x)$ and the input $f(x)$ for each γ respectively, such that

$$U^\gamma(x) = \begin{cases} 1 , & \text{when } u(x) > \gamma \\ 0 , & \text{when } u(x) \leq \gamma \end{cases} , \quad x \in \Omega , \quad i = 1, \ldots, n ; \tag{1.4}$$

and $|\partial U^\gamma|$ denotes the perimeter of U^γ and $\left| U^\gamma \triangle F^\gamma \right|$ the area of the symmetric difference of the two level sets, respectively.

Yin et al. [30] pointed out that minimizing such a layer-wise energy function (1.3) actually amounts to properly stacking the optimal U^γs, which corresponds to solving (1.2) for each given binary indicator function of F^γ. In other words, solving (1.1) can be reduced to optimizing a sequence of binary constrained problems as (1.2). Since $U^{\gamma_1} \subset U^{\gamma_2}$ when $\gamma_1 \geq \gamma_2$, the process recovers the optimum $u^*(x)$ of (1.1) by properly arranging all the associated level sets U^γ, $\gamma \in (-\infty, +\infty)$. The same result was also discovered by Darbon et al. [10,11] in an image graph setting, where the anisotropic total-variation term was considered and a fast graph-cut based algorithm introduced. Goldfarb and Yin also developed an efficient pre-flow based graph-cut approach to such L1 image approximation regularized by discretized total-variation.