

A Posteriori Error Estimates of Lowest Order Raviart-Thomas Mixed Finite Element Methods for Bilinear Optimal Control Problems

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Abstract. A Raviart-Thomas mixed finite element discretization for general bilinear optimal control problems is discussed. The state and co-state are approximated by lowest order Raviart-Thomas mixed finite element spaces, and the control is discretized by piecewise constant functions. *A posteriori* error estimates are derived for both the coupled state and the control solutions, and the error estimators can be used to construct more efficient adaptive finite element approximations for bilinear optimal control problems. An adaptive algorithm to guide the mesh refinement is also provided. Finally, we present a numerical example to demonstrate our theoretical results.

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1. Introduction

Optimal control problems with various physical backgrounds arise in many practical scientific areas, and efficient numerical methods can substantially reduce associated computational work. Two early papers devoted to finite element methods for linear elliptic optimal control problems studied error estimates for the numerical discretization [14, 15], the finite element approach for a class of constrained nonlinear optimal control problems has been considered [16], and *a posteriori* error estimates for the finite element approximation of nonlinear elliptic optimal control problems have also been derived [21]. Adaptive

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finite element methods are now widely used in engineering simulations. Efficient adaptive finite element methods can greatly reduce discretization errors, and there have been recent developments for optimal control problems. Some basic concepts have been introduced for adaptive finite element discretization in optimal control problems involving partial differential equations [5], and *a posteriori* error estimates for distributed elliptic optimal control problems have been obtained [19]. Recently, Feng has discussed an adaptive finite element method for the estimation of distributed parameters in elliptic equations [13]. All of this work addresses the standard finite element method.

In many control problems, the objective functional contains the gradient of the state variables, which should be rendered accurately in the numerical discretization of the coupled state equations. Mixed finite element methods are appropriate for the state equations in such cases, since both the scalar variable and its flux variable can then be approximated to the same accuracy. Many important contributions have been made to aspects of the mixed finite element method for linear optimal control problems, but there is not much theoretical analysis of mixed finite element approximations for bilinear or strong nonlinear optimal control problems. However, *a priori* error estimates and superconvergence for linear optimal control problems using mixed finite element methods have been obtained [9, 11, 12, 23], and also *a posteriori* error estimates of mixed finite element methods for quadratic elliptic optimal control problems [10].

The mixed finite element approximation for quadratic optimal control problems governed by semilinear elliptic equation has previously been discussed, including *a posteriori* error estimates for the mixed finite element solution [22]. In this paper, we consider adaptive mixed finite element methods for bilinear optimal control problems. We adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω , with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p = \sum_{|\alpha|\leq m} \|D^\alpha v\|_{L^p(\Omega)}^p$ and a semi-norm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p=2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$.

The general form of the bilinear optimal control problems of interest is

$$\min_{u \in KCU} \{g_1(\mathbf{p}) + g_2(y) + j(u)\} \quad (1.1)$$

subject to the state equation

$$\operatorname{div} \mathbf{p} + yu = f, \quad x \in \Omega, \quad (1.2)$$

$$\mathbf{p} = -A\nabla y, \quad x \in \Omega, \quad (1.3)$$

$$y = 0, \quad x \in \partial\Omega, \quad (1.4)$$

where $\Omega \subset \mathbf{R}^2$ is a bounded open set with boundary $\partial\Omega$ and $f \in L^2(\Omega)$. The 2×2 coefficient matrix $A(x) = (a_{i,j}(x))_{2 \times 2} \in L^\infty(\Omega; \mathbf{R}^{2 \times 2})$ is symmetric, and there is a constant $c > 0$ such that $\mathbf{X}'A\mathbf{X} \geq c\|\mathbf{X}\|_{\mathbf{R}^2}^2$ for any vector $\mathbf{X} \in \mathbf{R}^2$. Furthermore, we assume that K is a closed

convex set in $U = L^2(\Omega)$ defined by

$$K = \{u \in L^2(\Omega) : u \geq 0 \text{ a.e. in } \Omega\},$$

and that g'_1, g'_2 , and j' are locally Lipschitz continuous — i.e.

$$\begin{aligned} |j'(v(x_1)) - j'(v(x_2))| &\leq C|x_1 - x_2|, & \forall v \in K, x_1, x_2 \in \bar{\Omega}; \\ |g'_1(\mathbf{p}_1) - g'_1(\mathbf{p}_2)| &\leq C|\mathbf{p}_1 - \mathbf{p}_2|, & \forall \mathbf{p}_1, \mathbf{p}_2 \in H(\operatorname{div}\Omega); \\ |g'_1(y_1) - g'_1(y_2)| &\leq C|y_1 - y_2|, & \forall y_1, y_2 \in L^2(\Omega). \end{aligned}$$

There exists a constant $c > 0$ such that

$$(j'(u_1) - j'(u_2), u_1 - u_2) \geq c\|u_1 - u_2\|^2, \quad \forall u_1, u_2 \in K,$$

and we also recall a result from Becker and Vexler [6]:

Lemma 1.1. *For every function $f \in L^2(\Omega)$, the solution y of*

$$-\operatorname{div}(A\nabla y) + yu = f \quad \text{in } \Omega, \quad y|_{\partial\Omega} = 0, \quad (1.5)$$

belongs to $H_0^1(\Omega) \cap H^2(\Omega)$. Moreover, there exists a positive constant C such that

$$\|y\|_{W^{2,2}(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \quad (1.6)$$

Due to Lemma 1.1, the state equations (1.2)-(1.3) admit a unique solution in $H_0^1(\Omega) \cap H^2(\Omega)$.

The plan of this paper is as follows. In Section 2, we present the lowest order Raviart-Thomas mixed finite element discretization for general bilinear optimal control problems. In Section 3, we obtain a posteriori error estimates for some intermediate errors, and derive *a posteriori* error estimates for the lowest order Raviart-Thomas mixed finite element method for the bilinear optimal control problems. Next, we introduce an adaptive algorithm to guide the mesh refinement in Section 4. A numerical example is provided in Section 5, and our concluding remarks and suggestions for future work are in Section 6.

2. Mixed Methods for Optimal Control Problems

The lowest order Raviart-Thomas mixed finite element discretization of the general bilinear optimal control problem (1.1)-(1.4) is discussed in this Section. We consider the Hilbert space

$$V = H(\operatorname{div}\Omega) = \{\mathbf{v} \in (L^2(\Omega))^2, \operatorname{div}\mathbf{v} \in L^2(\Omega)\}, \quad W = L^2(\Omega),$$

with the norm

$$\|\mathbf{v}\|_{H(\operatorname{div}\Omega)} = \|\mathbf{v}\|_{\operatorname{div}} = (\|\mathbf{v}\|_{0,\Omega}^2 + \|\operatorname{div}\mathbf{v}\|_{0,\Omega}^2)^{1/2},$$

to first recast (1.1)-(1.4) in the following weak form: find $(\mathbf{p}, y, u) \in \mathbf{V} \times W \times U$ such that

$$\min_{u \in K \subset U} \{g_1(\mathbf{p}) + g_2(y) + j(u)\} \quad (2.1)$$

$$(A^{-1}\mathbf{p}, \mathbf{v}) - (y, \operatorname{div} \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \quad (2.2)$$

$$(\operatorname{div} \mathbf{p}, w) + (yu, w) = (f, w), \quad \forall w \in W, \quad (2.3)$$

where the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^2$ is indicated by (\cdot, \cdot) . For ease of exposition, we will assume that Ω is a polygon; and letting \mathcal{T}_h be a regular triangulation of Ω , we assume the angle condition is satisfied — i.e. there is a positive constant C such that for all $T \in \mathcal{T}_h$ we have $C^{-1}h_T^2 \leq |T| \leq Ch_T^2$, where $|T|$ is the area of T and h_T is the diameter of T . We write $h = \max h_T$, and C (or c) to denote a general positive constant independent of h .

It is well known (e.g. see [10]) that the optimal control problem (2.1)-(2.3) has at least one solution (\mathbf{p}^*, y^*, u^*) ; and that if triplet (\mathbf{p}^*, y^*, u^*) is the solution of (2.1)-(2.3), then there is a co-state $(\mathbf{q}^*, z^*) \in \mathbf{V} \times W$ such that $(\mathbf{p}^*, y^*, \mathbf{q}^*, z^*, u^*)$ satisfies the following optimality conditions:

$$(A^{-1}\mathbf{p}^*, \mathbf{v}) - (y^*, \operatorname{div} \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \quad (2.4)$$

$$(\operatorname{div} \mathbf{p}^*, w) + (y^* u^*, w) = (f, w), \quad \forall w \in W, \quad (2.5)$$

$$(A^{-1}\mathbf{q}^*, \mathbf{v}) - (z^*, \operatorname{div} \mathbf{v}) = -(g'_1(\mathbf{p}^*), \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}, \quad (2.6)$$

$$(\operatorname{div} \mathbf{q}^*, w) + (z^* u^*, w) = (g'_2(y^*), w), \quad \forall w \in W, \quad (2.7)$$

$$(j'(u^*) - y^* z^*, \tilde{u} - u^*) \geq 0, \quad \forall \tilde{u} \in K, \quad (2.8)$$

where g'_1 , g'_2 , and j' are the derivatives of g_1 , g_2 , and j . Henceforth, we simply write the product as (\cdot, \cdot) wherever there should be no confusion.

Let $\mathbf{V}_h \times W_h \subset \mathbf{V} \times W$ denote the Raviart-Thomas mixed finite element space [25] of the lowest order associated with the triangulation \mathcal{T}_h of $\bar{\Omega}$ — viz.

$$\mathbf{V}(T) = \{\mathbf{v} \in P_0^2(T) + x \cdot P_0(T)\}, \quad W(T) = P_0(T), \quad \forall T \in \mathcal{T}_h,$$

where $P_0(T)$ denotes the piecewise constant space, $x = (x_1, x_2)$ is treated as a vector, and

$$\mathbf{V}_h := \{\mathbf{v}_h \in \mathbf{V} : \forall T \in \mathcal{T}_h, \mathbf{v}_h|_T \in \mathbf{V}(T)\},$$

$$W_h := \{w_h \in W : \forall T \in \mathcal{T}_h, w_h|_T \in W(T)\}.$$

Associated with \mathcal{T}_h is another finite-dimensional subspace of K — viz.

$$K_h := \{\tilde{u}_h \in K : \forall T \in \mathcal{T}_h, \tilde{u}_h|_T = \text{constant}\}.$$

The mixed finite element discretization of (2.1)-(2.3) is then as follows.

Compute $(\mathbf{p}_h, y_h, u_h) \in \mathbf{V}_h \times W_h \times K_h$ such that

$$\min_{u_h \in K_h} \{g_1(\mathbf{p}_h) + g_2(y_h) + j(u_h)\} \quad (2.9)$$

$$(A^{-1}\mathbf{p}_h, \mathbf{v}_h) - (y_h, \operatorname{div} \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \quad (2.10)$$

$$(\operatorname{div} \mathbf{p}_h, w_h) + (y_h u_h, w_h) = (f, w_h), \quad \forall w_h \in W_h. \quad (2.11)$$

The optimal control problem (2.9)-(2.11) again has at least the triplet solution $(\mathbf{p}_h^*, y_h^*, u_h^*)$ and the co-state $(\mathbf{q}_h^*, z_h^*) \in \mathbf{V}_h \times W_h$, where $(\mathbf{p}_h^*, y_h^*, \mathbf{q}_h^*, z_h^*, u_h^*)$ satisfies the optimality conditions

$$(A^{-1}\mathbf{p}_h^*, \mathbf{v}_h) - (y_h^*, \operatorname{div} \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \quad (2.12)$$

$$(\operatorname{div} \mathbf{p}_h^*, w_h) + (y_h^* u_h^*, w_h) = (f, w_h), \quad \forall w_h \in W_h, \quad (2.13)$$

$$(A^{-1}\mathbf{q}_h^*, \mathbf{v}_h) - (z_h^*, \operatorname{div} \mathbf{v}_h) = -(g'_1(\mathbf{p}_h^*), \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \quad (2.14)$$

$$(\operatorname{div} \mathbf{q}_h^*, w_h) + (z_h^* u_h^*, w_h) = (g'_2(y_h^*), w_h), \quad \forall w_h \in W_h, \quad (2.15)$$

$$(j'(u_h^*) - y_h^* z_h^*, \tilde{u}_h - u_h^*) \geq 0, \quad \forall \tilde{u}_h \in K_h. \quad (2.16)$$

Let \mathcal{E}_h denote the set of side elements in \mathcal{T}_h . The local mesh size h is defined on both \mathcal{T}_h and \mathcal{E}_h by $h|_T := h_T$ for $T \in \mathcal{T}_h$ and $h|_E := h_E$ for $E \in \mathcal{E}_h$, respectively. For all $E \in \mathcal{E}_h$, we fix one direction of a unit normal on $E \subset \partial\Omega$ pointing outwards from Ω . We define an operator $[v] : H^1(\mathcal{T}_h) \rightarrow L^2(\mathcal{E}_h)$ for the jump of the function v across the edge E , and let \mathbf{t} denote the tangential unit vector along E . We also define $S^0(\mathcal{T}_h) \subset L^2(\Omega)$ as the piecewise constant space and $S^1(\mathcal{T}_h) \subset H^1(\Omega)$ or $S_0^1(\mathcal{T}_h) \subset H_0^1(\Omega)$ as continuous and piecewise linear functions, where piecewise is understood to be with respect to \mathcal{T}_h . We consider Clement's interpolation operator $I_h : H^1(\Omega) \rightarrow S^1(\mathcal{T}_h)$ that satisfies [5]

$$\|v - I_h v\|_{0,T} \leq Ch_T \|v\|_{1,w_T}, \quad \forall v \in H_0^1(\Omega), \quad (2.17)$$

$$\|v - I_h v\|_{0,E} \leq Ch_E^{1/2} \|v\|_{1,w_E}, \quad \forall v \in H_0^1(\Omega), \quad (2.18)$$

for each $T \in \mathcal{T}_h$ and $E \in \mathcal{E}_h$, where $w_T = \{T' \in \mathcal{T}_h, \bar{T} \cap \bar{T}' \neq \emptyset\}$ and $w_E = \{T \in \mathcal{T}_h, E \in \bar{T}\}$.

Next we define the standard $L^2(\Omega)$ -orthogonal projection $P_h : W \rightarrow W_h$, which satisfies the approximation property [3]

$$\|h^{-1} \cdot (v - P_h v)\|_0 \leq C \|\nabla_h v\|_0, \quad \forall v \in H^1(\mathcal{T}_h); \quad (2.19)$$

and also the interpolation operator $\Pi_h : \mathbf{V} \rightarrow \mathbf{V}_h$, which for any $\mathbf{q} \in \mathbf{V}$ satisfies

$$\int_T (\mathbf{q} - \Pi_h \mathbf{q}) \cdot \mathbf{v}_h dx dy = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, T \in \mathcal{T}_h.$$

We have the commuting diagram property

$$\operatorname{div} \circ \Pi_h = P_h \circ \operatorname{div} : \mathbf{V} \rightarrow W_h \quad \text{and} \quad \operatorname{div}(I - \Pi_h)\mathbf{V} \perp W_h, \quad (2.20)$$

wherein and hereafter I denotes the identity operator and the interpolation operator Π_h satisfies a local error estimate

$$\|h^{-1} \cdot (\mathbf{q} - \Pi_h \mathbf{q})\|_0 \leq C |\mathbf{q}|_{1, \mathcal{T}_h}, \quad \mathbf{q} \in H^1(\mathcal{T}_h) \cap \mathbf{V}. \quad (2.21)$$

Let us now introduce a new hypothesis — viz. A is element-wise smooth and $A \in C^{1,0}(\bar{\Omega})^{2 \times 2}$, implying that

$$-\operatorname{div}(A\nabla \cdot) : H_0^1(\Omega) \cap H^2(\Omega) \rightarrow L^2(\Omega) \text{ is invertible,}$$

and there exists a constant $C > 0$ such that for all $v \in H_0^1(\Omega)$ and $\operatorname{div}(A\nabla v) \in L^2(\Omega)$ we have

$$\|v\|_{2, \mathcal{T}_h} \leq C \|\operatorname{div}(A\nabla v)\|_0.$$

3. A Posteriori Error Estimates

Given fixed $u \in K$, let L_1 and L_2 be the inverse operators of the state equation (2.3) such that $\mathbf{p}(u) = L_1 B u$ and $y(u) = L_2 B u$ are the solutions of (2.3). Similarly, for given $u_h \in K_h$, $\mathbf{p}_h(u_h) = L_{1h} B u_h$, $y_h(u_h) = L_{2h} B u_h$ are the solutions of the discrete state equation (2.11). Let

$$\begin{aligned} J(u) &= g_1(L_1 B u) + g_2(L_2 B u) + j(u), \\ J_h(u_h) &= g_1(L_{1h} B u_h) + g_2(L_{2h} B u_h) + j(u_h). \end{aligned}$$

Clearly, J and J_h are well defined and continuous on K and K_h , and the functional J_h can be extended naturally on K , so (2.1) and (2.9) can be represented as

$$\min_{u \in K} \{J(u)\}, \quad (3.1)$$

$$\min_{u_h \in K_h} \{J_h(u_h)\}. \quad (3.2)$$

An additional assumption is needed. We assume that the cost function J is strictly convex near the solution u^* . Thus for the solution u^* there exists a neighbourhood of u^* in L^2 such that J is convex, in the sense that there is a constant $c > 0$ satisfying

$$(J'(u^*) - J'(v), u^* - v) \geq c \|u^* - v\|_0^2, \quad (3.3)$$

for all v in this neighbourhood of u^* . The convexity of $J(\cdot)$ is closely related to the second order sufficient optimality conditions of optimal control problems, assumed in many studies on their numerical solution. We shall assume the above inequality throughout this paper — cf. also [1, 2]. We are now able to derive a main result.

Lemma 3.1. Let u^* and u_h^* be the solutions of (3.1) and (3.2), respectively. In addition, assume that $(J'_h(u_h^*))|_T \in H^1(T)$, $\forall T \in \mathcal{T}_h$, and that there is a $v_h \in K_h$ such that

$$|(J'_h(u_h^*), v_h - u^*)| \leq C \sum_{T \in \mathcal{T}_h} h_T \|J'_h(u_h^*)\|_{H^1(T)} \cdot \|u^* - u_h^*\|_{L^2(T)}. \quad (3.4)$$

Then we have

$$\|u^* - u_h^*\|_0^2 \leq C\eta_1^2 + C\|z(u_h^*) - z_h^*\|_0^2 + C\|y(u_h^*) - y_h^*\|_0^2, \quad (3.5)$$

where

$$\eta_1^2 = \sum_{T \in \mathcal{T}_h} h_T^2 \|j'(u_h^*) - y_h^* z_h^*\|_{H^1(T)}^2. \quad (3.6)$$

Proof. It follows from (3.1) and (3.2) that

$$(J'(u^*), u^* - v) \leq 0, \quad \forall v \in K, \quad (3.7)$$

$$(J'_h(u_h^*), u_h^* - v_h) \leq 0, \quad \forall v_h \in K_h \subset K. \quad (3.8)$$

Hence from (3.3) and (3.7)-(3.8) we have

$$\begin{aligned} c\|u^* - u_h^*\|_0^2 &\leq (J'(u^*) - J'(u_h^*), u^* - u_h^*) \\ &\leq -(J'(u_h^*), u^* - u_h^*) \\ &= (J'_h(u_h^*), u_h^* - u^*) + (J'_h(u_h^*) - J'(u_h^*), u^* - u_h^*) \\ &\leq (J'_h(u_h^*), v_h - u^*) + (J'_h(u_h^*) - J'(u_h^*), u^* - u_h^*); \end{aligned} \quad (3.9)$$

and from (3.4), (3.9) and the Schwartz inequality that

$$\begin{aligned} c\|u^* - u_h^*\|_0^2 &\leq C \sum_{T \in \mathcal{T}_h} h_T \|J'_h(u_h^*)\|_{H^1(T)} \|u^* - u_h^*\|_{L^2(T)} \\ &\quad + C \|J'_h(u_h^*) - J'(u_h^*)\|_0 \|u^* - u_h^*\|_0 \\ &\leq C(\delta) \sum_{T \in \mathcal{T}_h} h_T^2 \|J'_h(u_h^*)\|_{H^1(T)}^2 \\ &\quad + C(\delta) \|J'_h(u_h^*) - J'(u_h^*)\|_0^2 + C\delta \|u^* - u_h^*\|_0^2, \end{aligned} \quad (3.10)$$

where δ is an arbitrary positive number. It is also not difficult to show that

$$J'_h(u_h^*) = j'(u_h^*) - y_h^* z_h^*, \quad J'(u_h^*) = j'(u_h^*) - y(u_h^*) z(u_h^*), \quad (3.11)$$

where $z(u_h^*)$ is the solution of the following equations with $\tilde{u} = u_h^*$:

$$(A^{-1} \mathbf{p}(\tilde{u}), \mathbf{v}) - (y(\tilde{u}), \operatorname{div} \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \quad (3.12)$$

$$(\operatorname{div} \mathbf{p}(\tilde{u}), w) + (y(\tilde{u}) \tilde{u}, w) = (f, w), \quad \forall w \in W, \quad (3.13)$$

$$(A^{-1} \mathbf{q}(\tilde{u}), \mathbf{v}) - (z(\tilde{u}), \operatorname{div} \mathbf{v}) = -(g'_1(\mathbf{p}(\tilde{u})), \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}, \quad (3.14)$$

$$(\operatorname{div} \mathbf{q}(\tilde{u}), w) + (z(\tilde{u}) \tilde{u}, w) = (g'_2(y(\tilde{u})), w), \quad \forall w \in W. \quad (3.15)$$

Further, from (3.11) it is easy to derive

$$\begin{aligned}
\|J'_h(u_h^*) - J'(u_h^*)\|_0 &= \|y_h^* z_h^* - y(u_h^*) z(u_h^*)\|_0 \\
&= \|y_h^* (z_h^* - z(u_h^*)) + (y_h^* - y(u_h^*)) z(u_h^*)\|_0 \\
&\leq \|y_h^* (z_h^* - z(u_h^*))\|_0 + \|(y_h^* - y(u_h^*)) z(u_h^*)\|_0 \\
&\leq \|y_h^*\|_{0,\infty} \cdot \|z_h^* - z(u_h^*)\|_0 + \|z(u_h^*)\|_{0,\infty} \cdot \|y_h^* - y(u_h^*)\|_0 \\
&\leq C \|z_h^* - z(u_h^*)\|_0 + C \|y_h^* - y(u_h^*)\|_0, \tag{3.16}
\end{aligned}$$

where $\|y_h^*\|_{0,\infty} \leq C$ and $\|z(u_h^*)\|_{0,\infty} \leq C$ have been used. It is clear that (3.5) can be derived from (3.10)-(3.16). \square

Let us now fix a function $u_h^* \in U_h$, and let $(\mathbf{p}(u_h^*), y(u_h^*)) \in \mathbf{V} \times W$ be the solution of the equations (3.12)-(3.13) with $\tilde{u} = u_h^*$. We also set some intermediate errors $\varepsilon_1 := \mathbf{p}(u_h^*) - \mathbf{p}_h^*$, $e_1 := y(u_h^*) - y_h^*$. To analyze this fixed u_h^* approach, let us first note the following error equations obtained from (2.10)-(2.11) and (3.12)-(3.13):

$$(A^{-1} \varepsilon_1, \mathbf{v}_h) - (e_1, \operatorname{div} \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{3.17}$$

$$(\operatorname{div} \varepsilon_1, w_h) + ((y(u_h^*) - y_h^*) u_h^*, w_h) = 0, \quad \forall w_h \in W_h. \tag{3.18}$$

It follows from the uniqueness of the solutions for (3.12)-(3.13) that $y(u_h^*) \in H_0^1(\Omega)$, and

$$\mathbf{p}(u_h^*) = -A \nabla y(u_h^*), \quad \forall x \in \Omega, \tag{3.19}$$

$$\operatorname{div} \mathbf{p}(u_h^*) + y(u_h^*) u_h^* = f, \quad \forall x \in \Omega. \tag{3.20}$$

In order to estimate $\|\mathbf{p}(u_h^*) - \mathbf{p}_h^*\|_{\operatorname{div}}$ and $\|y(u_h^*) - y_h^*\|_0$, we need *a priori* regularity estimates for the auxiliary problem

$$-\operatorname{div}(A \nabla \xi) + \xi u_h^* = G, \quad x \in \Omega, \quad \xi|_{\partial\Omega} = 0. \tag{3.21}$$

The following lemma provides the desired *a priori* estimate (e.g. see [6]).

Lemma 3.2. *Let ξ be the solution of (3.21) and assume that Ω is convex. Then*

$$\|\xi\|_2 \leq C \|G\|_0. \tag{3.22}$$

We can now prove:

Lemma 3.3. *For the Raviart-Thomas elements there is a positive constant C , dependent only on A , Ω and the shape of the elements, such that*

$$\|\mathbf{p}(u_h^*) - \mathbf{p}_h^*\|_{\operatorname{div}} + \|y(u_h^*) - y_h^*\|_0 \leq C \eta_2, \tag{3.23}$$

where

$$\begin{aligned}
\eta_2 := \left(\sum_{T \in \mathcal{T}_h} \eta_{2T}^2 \right)^{1/2} &:= \left[\sum_{T \in \mathcal{T}_h} \left(\|f - \operatorname{div} \mathbf{p}_h^* - y_h^* u_h^*\|_{0,T}^2 + h_T^2 \cdot \|\operatorname{Curl}_h(A^{-1} \mathbf{p}_h^*)\|_{0,T}^2 \right. \right. \\
&\quad \left. \left. + \|h_E^{1/2} [(A^{-1} \mathbf{p}_h^*) \cdot \mathbf{t}]\|_{0,\partial T}^2 + h_T^2 \cdot \|A^{-1} \mathbf{p}_h^*\|_{0,T}^2 \right) \right]^{1/2}. \tag{3.24}
\end{aligned}$$

Proof. We analyze a Helmholtz decomposition [8] of $A^{-1}\mathbf{p}_h^*$ with a fixed $\varphi \in H_0^1(\Omega)$ such that $-\operatorname{div}(A\nabla\varphi) = \operatorname{div}\mathbf{p}_h^*$. Then there is some $\psi \in H^1(\Omega)$ satisfying $\int_{\Omega} \psi dx = 0$, $\operatorname{Curl}\psi \perp \nabla H_0^1(\Omega)$ and

$$\mathbf{p}_h^* = -A\nabla\varphi + \operatorname{Curl}\psi. \quad (3.25)$$

From (3.19) and (3.25) we derive

$$\varepsilon_1 = A\nabla\chi - \operatorname{Curl}\psi \quad \text{with} \quad \chi = \varphi - y(u_h^*) \in H_0^1(\Omega), \quad (3.26)$$

and hence the error decomposition

$$(A^{-1}\varepsilon_1, \varepsilon_1) = (A\nabla\chi, \nabla\chi) + (A^{-1}\operatorname{Curl}\psi, \operatorname{Curl}\psi). \quad (3.27)$$

It follows from the Poincare inequality and (2.19) that

$$\begin{aligned} (A\nabla\chi, \nabla\chi) &= (\nabla\chi, \varepsilon_1) = -(\operatorname{div}\varepsilon_1, \chi) \\ &= (\operatorname{div}\varepsilon_1, P_h\chi - \chi) - (\operatorname{div}\varepsilon_1, P_h\chi) \\ &\leq C\|h_T \cdot \operatorname{div}\varepsilon_1\|_0 \cdot \|A^{1/2}\nabla\chi\|_0 + C\|\operatorname{div}\varepsilon_1\|_0 \cdot \|P_h\chi\|_0 \\ &\leq C\|h_T \cdot \operatorname{div}\varepsilon_1\|_0 \cdot \|A^{1/2}\nabla\chi\|_0 + C\|\operatorname{div}\varepsilon_1\|_0 \cdot \|A^{1/2}\nabla\chi\|_0. \end{aligned} \quad (3.28)$$

To estimate the second contribution to the right-hand side of (3.27), we utilize Clement's operator I_h . Note that $I_h\psi \in S^1(\mathcal{T}_h) \subset H^1(\Omega)$, $\operatorname{Curl}I_h\psi \in S^0(\mathcal{T}_h)^2 \cap H^1(\operatorname{div}\Omega) \subset \mathbf{V}_h$ and $\operatorname{Curl}I_h\psi \perp \nabla H_0^1(\Omega)$, and of course $\operatorname{div}(\operatorname{Curl}I_h\psi) = 0$. Thus we obtain

$$(A^{-1}\operatorname{Curl}\psi, \operatorname{Curl}I_h\psi) = -(A^{-1}\varepsilon_1, \operatorname{Curl}I_h\psi) = -(e_1, \operatorname{div}\operatorname{Curl}I_h\psi) = 0.$$

From (3.25) and (2.17)-(2.18) we infer

$$\begin{aligned} (A^{-1}\operatorname{Curl}\psi, \operatorname{Curl}\psi) &= (A^{-1}\operatorname{Curl}\psi, \operatorname{Curl}(\psi - I_h\psi)) = (A^{-1}\mathbf{p}_h^*, \operatorname{Curl}(\psi - I_h\psi)) \\ &= -(\psi - I_h\psi, \operatorname{Curl}_h(A^{-1}\mathbf{p}_h^*)) + ([(A^{-1}\mathbf{p}_h^*) \cdot \mathbf{t}], \psi - I_h\psi)_{\mathcal{E}_h} \\ &\leq C(\|h_T \cdot \operatorname{Curl}_h(A^{-1}\mathbf{p}_h^*)\|_0 + \|h_E^{1/2} \cdot [(A^{-1}\mathbf{p}_h^*) \cdot \mathbf{t}]\|_{0, \mathcal{E}_h})\|\psi\|_1, \end{aligned} \quad (3.29)$$

and from Poincare's inequality deduce that

$$\|\psi\|_1 \leq C\|\nabla\psi\|_0 = C\|\operatorname{Curl}\psi\|_0 \leq C\|A^{-1/2}\operatorname{Curl}\psi\|_0. \quad (3.30)$$

From (3.20) we have

$$\begin{aligned} \operatorname{div}\varepsilon_1 &= f - \operatorname{div}\mathbf{p}_h^* - y(u_h^*)u_h^* \\ &= f - \operatorname{div}\mathbf{p}_h^* - y_h^*u_h^* - (y(u_h^*) - y_h^*)u_h^* \\ &= f - \operatorname{div}\mathbf{p}_h^* - y_h^*u_h^* - e_1 \cdot u_h^*, \end{aligned} \quad (3.31)$$

whence

$$\begin{aligned} \|\operatorname{div}\varepsilon_1\| &= \|f - \operatorname{div}\mathbf{p}_h^* - y_h^*u_h^* - e_1 \cdot u_h^*\| \\ &\leq \|f - \operatorname{div}\mathbf{p}_h^* - y_h^*u_h^*\|_0 + \|e_1\|_0 \cdot \|u_h^*\|_0 \\ &\leq C(\|f - \operatorname{div}\mathbf{p}_h^* - y_h^*u_h^*\|_0 + \|e_1\|_0), \end{aligned} \quad (3.32)$$

so from (3.27)-(3.30) we have

$$\begin{aligned} \|\varepsilon_1\|_{\text{div}} \leq C & \left(\|f - \text{div} \mathbf{p}_h^* - y_h^* u_h^*\|_0 + \|e_1\|_0 \right. \\ & \left. + h_T \cdot \|\text{Curl}_h(A^{-1} \mathbf{p}_h^*)\|_0 + \|h_E^{1/2} [(A^{-1} \mathbf{p}_h^*) \cdot \mathbf{t}]\|_{0, \mathcal{E}_h} \right). \end{aligned} \quad (3.33)$$

Let us now estimate $\|e_1\|_0$. If ξ denotes the solution of (3.21) with $G = y(u_h^*) - y_h^*$, it follows from (2.12)-(2.13), (2.20), (3.19)-(3.20) and (3.21) that

$$\begin{aligned} \|e_1\|_0^2 &= (y(u_h^*) - y_h^*, -\text{div}(A\nabla\xi) + \xi u_h^*) \\ &= -(\mathbf{p}(u_h^*), \nabla\xi) + (y_h^*, \text{div} \Pi_h(A\nabla\xi)) + ((y(u_h^*) - y_h^*), \xi u_h^*) \\ &= (\text{div} \mathbf{p}(u_h^*), \xi) + (A^{-1} \mathbf{p}_h^*, \Pi_h(A\nabla\xi)) + ((y(u_h^*) - y_h^*) u_h^*, \xi) \\ &= (f - \text{div} \mathbf{p}_h^* - y(u_h^*) u_h^*, \xi) + ((y(u_h^*) - y_h^*) u_h^*, \xi) - (A^{-1} \mathbf{p}_h^*, (I - \Pi_h)(A\nabla\xi)) \\ &= (f - \text{div} \mathbf{p}_h^* - y_h^* u_h^*, \xi) - (A^{-1} \mathbf{p}_h^*, (I - \Pi_h)(A\nabla\xi)) \\ &\leq C (\|f - \text{div} \mathbf{p}_h^* - y_h^* u_h^*\|_0 + \|h \cdot (A^{-1} \mathbf{p}_h^*)\|_0) \cdot \|\xi\|_2 \\ &\leq C(\delta) (\|f - \text{div} \mathbf{p}_h^* - y_h^* u_h^*\|_0^2 + \|h \cdot (A^{-1} \mathbf{p}_h^*)\|_0^2) + C\delta \|e_1\|_0^2, \end{aligned}$$

for any $w_h \in W_h$. Using the triangle inequality, one obtains

$$\|e_1\|_0 \leq C (\|f - \text{div} \mathbf{p}_h^* - y_h^* u_h^*\|_0 + \|h \cdot (A^{-1} \mathbf{p}_h^*)\|_0), \quad (3.34)$$

so Lemma 3.3 is proven by combining (3.33) and (3.34). \square

Moreover, the reverse inequality of (3.23) holds.

Lemma 3.4. *For the Raviart-Thomas elements there is a positive constant C , dependent only on A , Ω and the shape of the elements, such that*

$$C\eta_2 \leq \|\mathbf{p}(u_h^*) - \mathbf{p}_h^*\|_{\text{div}} + \|y(u_h^*) - y_h^*\|_0. \quad (3.35)$$

Proof. From (3.31) we obtain

$$f - \text{div} \mathbf{p}_h^* - y_h^* u_h^* = \text{div} \varepsilon_1 + e_1 \cdot u_h^*, \quad (3.36)$$

such that

$$\begin{aligned} \|f - \text{div} \mathbf{p}_h^* - y_h^* u_h^*\|_{0,T} &= \|\text{div} \varepsilon_1 + e_1 \cdot u_h^*\|_{0,T} \\ &\leq \|\varepsilon_1\|_{H(\text{div};T)} + \|e_1\|_{0,T} \cdot \|u_h^*\|_{0,T} \\ &\leq C (\|\varepsilon_1\|_{H(\text{div};T)} + \|e_1\|_{0,T}). \end{aligned} \quad (3.37)$$

Next, using the standard Bubble function technique, we fix $\varrho_T \in P_3$ with $0 \leq \varrho_T \leq 1 = \max \varrho_T$ and zero boundary values on T to derive

$$C \|\text{Curl}(A^{-1} \mathbf{p}_h^*)\|_{0,T}^2 \leq \|\varrho_T^{1/2} \cdot \text{Curl}(A^{-1} \mathbf{p}_h^*)\|_{0,T}^2. \quad (3.38)$$

Using (3.25) and (3.26), we obtain

$$\begin{aligned} \|\varrho_T^{1/2} \cdot \text{Curl}(A^{-1}\mathbf{p}_h^*)\|_{0,T}^2 &= \int_T (A^{-1}\varepsilon_1) \cdot \text{Curl}(\varrho_T^{1/2} \cdot \text{Curl}(A^{-1}\mathbf{p}_h^*)) dx \\ &\leq C \|A^{-1}\varepsilon_1\|_{0,T} \cdot |\varrho_T^{1/2} \cdot \text{Curl}(A^{-1}\mathbf{p}_h^*)|_{1,T} \\ &\leq C \|\varepsilon_1\|_{H(\text{div};T)} \cdot h_T^{-1} \cdot \|\varrho_T^{1/2} \cdot \text{Curl}(A^{-1}\mathbf{p}_h^*)\|_{0,T}, \end{aligned} \quad (3.39)$$

since $\varrho_T^{1/2} \cdot \text{Curl}(A^{-1}\mathbf{p}_h^*) \in P_{l+2}$ with zero boundary values on T . Combining (3.38) and (3.39) we have

$$h_T \cdot \|\text{Curl}(A^{-1}\mathbf{p}_h^*)\|_{0,T} \leq C \|\varepsilon_1\|_{H(\text{div};T)}. \quad (3.40)$$

Now, let ϱ_E denote the continuous function satisfying $\varrho_E \in P_2$ with $0 \leq \varrho_E \leq 1 = \max \varrho_E$ on w_E , and let $\sigma = [(A^{-1}\mathbf{p}_h^*) \cdot \mathbf{t}]$. Using continuous extension on the reference element [26], there exists an extension operator $P : C(E) \rightarrow C(w_E)$ satisfying $P\sigma|_E = \sigma$ and

$$c_1 h_E^{1/2} \|\sigma\|_{0,E} \leq \|\varrho_E^{1/2} P\sigma\|_{0,w_E} \leq c_2 h_E^{1/2} \|\sigma\|_{0,E}, \quad (3.41)$$

where c_1 and c_2 are positive constants. From integration by parts and (3.40)-(3.41) we obtain

$$\begin{aligned} C \|\sigma\|_{0,E}^2 &\leq \|\varrho_E^{1/2} \sigma\|_{0,E}^2 = - \int_E (\varrho_E P\sigma) \cdot [A^{-1}\varepsilon_1 \cdot \mathbf{t}] ds \\ &= - \int_{w_E} (A^{-1}\varepsilon_1) \cdot \text{Curl}(\varrho_E P\sigma) dx - \int_{w_E} (\varrho_E P\sigma) \text{Curl}(A^{-1}\varepsilon_1) dx \\ &= - \int_{w_E} (A^{-1}\varepsilon_1) \cdot \text{Curl}(\varrho_E P\sigma) dx - \int_{w_E} (\varrho_E P\sigma) \text{Curl}(A^{-1}\mathbf{p}_h^*) dx \\ &\leq \|\varepsilon_1\|_{0,w_E} \cdot |\varrho_E P\sigma|_{1,w_E} + \|\varrho_E P\sigma\|_{0,w_E} \cdot \|\text{Curl}(A^{-1}\mathbf{p}_h^*)\|_{0,w_E} \\ &\leq C h_E^{-1/2} \|\sigma\|_{0,E} \cdot \|\varepsilon_1\|_{H(\text{div};w_E)}, \end{aligned} \quad (3.42)$$

where the inverse estimate has been used. Consequently,

$$\|h^{1/2} [(A^{-1}\mathbf{p}_h^*) \cdot \mathbf{t}]\|_{0,E} \leq C \|\varepsilon_1\|_{H(\text{div};w_E)}. \quad (3.43)$$

Finally, as in (3.38), from integration by parts and (3.19) we have

$$\begin{aligned} C \|A^{-1}\mathbf{p}_h^*\|_{0,T}^2 &\leq \|\varrho_T^{1/2} (A^{-1}\mathbf{p}_h^*)\|_{0,T}^2 \\ &= - \int_T \varrho_T A^{-1}\varepsilon_1 (A^{-1}\mathbf{p}_h^*) dx - \int_T e_1 \text{div}(\varrho_T (A^{-1}\mathbf{p}_h^*)) dx \\ &\leq \|A^{-1}\varepsilon_1\|_{0,T} \cdot \|\varrho_T (A^{-1}\mathbf{p}_h^*)\|_{0,T} + \|e_1\|_{0,T} \cdot |\varrho_T (A^{-1}\mathbf{p}_h^*)|_{1,T} \\ &\leq (\|A^{-1}\varepsilon_1\|_{0,T} + \|e_1\|_{0,T} \cdot h_T^{-1}) \cdot \|\varrho_T (A^{-1}\mathbf{p}_h^*)\|_{0,T}, \end{aligned} \quad (3.44)$$

where the inverse inequality has been used. From (3.44) it is clear that

$$h_T \cdot \|A^{-1} \mathbf{p}_h^*\|_{0,T} \leq C \left(\|e_1\|_{0,T} + h_T \cdot \|A^{-1} \varepsilon_1\|_{0,T} \right), \quad (3.45)$$

so Lemma 3.4 is proved on combining (3.37), (3.40), (3.43) and (3.45). \square

Arguing as in the proof of Lemma 3.3 and using Lemma 3.2, we thus obtain the following results.

Lemma 3.5. *For the Raviart-Thomas elements there is a positive constant C , dependent only on A , Ω and the shape of the elements, such that*

$$\|\mathbf{q}(u_h^*) - \mathbf{q}_h^*\|_{\text{div}} + \|z(u_h^*) - z_h^*\|_0 \leq C(\eta_2 + \eta_3), \quad (3.46)$$

where

$$\begin{aligned} \eta_3 := \left(\sum_{T \in \mathcal{T}_h} \eta_{3T}^2 \right)^{1/2} &:= \left[\sum_{T \in \mathcal{T}_h} \left(\|g_2'(y_h^*) - \text{div} \mathbf{q}_h^* - z_h^* u_h^*\|_{0,T}^2 + h_T^2 \cdot \|\text{Curl}_h(A^{-1} \mathbf{q}_h^* + g_1'(\mathbf{p}_h^*))\|_{0,T}^2 \right. \right. \\ &\quad \left. \left. + h_E^{1/2} \|(A^{-1} \mathbf{q}_h^* + g_1'(\mathbf{p}_h^*)) \cdot \mathbf{t}\|_{0,\partial T}^2 + h_T^2 \cdot \|A^{-1} \mathbf{q}_h^* + g_1'(\mathbf{p}_h^*)\|_{0,T}^2 \right) \right]^{1/2}. \end{aligned} \quad (3.47)$$

In terms of intermediate errors, we can decompose the errors as

$$\begin{aligned} \mathbf{p}^* - \mathbf{p}_h^* &= \mathbf{p}^* - \mathbf{p}(u_h^*) + \mathbf{p}(u_h^*) - \mathbf{p}_h^* := \epsilon_1 + \varepsilon_1, \\ y^* - y_h^* &= y^* - y(u_h^*) + y(u_h^*) - y_h^* := r_1 + e_1, \\ \mathbf{q}^* - \mathbf{q}_h^* &= \mathbf{q}^* - \mathbf{q}(u_h^*) + \mathbf{q}(u_h^*) - \mathbf{q}_h^* := \epsilon_2 + \varepsilon_2, \\ z^* - z_h^* &= z^* - z(u_h^*) + z(u_h^*) - z_h^* := r_2 + e_2, \end{aligned}$$

and from standard results in mixed finite element methods [4] we have the following results.

Lemma 3.6. *There is a positive constant C independent of h such that*

$$\|\epsilon_1\|_{\text{div}} + \|r_1\|_0 \leq C \|u^* - u_h^*\|_0, \quad (3.48)$$

$$\|\epsilon_2\|_{\text{div}} + \|r_2\|_0 \leq C \|u^* - u_h^*\|_0. \quad (3.49)$$

Proof. From (2.4)-(2.7) and (3.12)-(3.15), we obtain the error equations

$$(A^{-1} \epsilon_1, \mathbf{v}) - (r_1, \text{div} \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \quad (3.50)$$

$$(\text{div} \epsilon_1, w) + (r_1 u^*, w) = -(y(u_h^*)(u^* - u_h^*), w), \quad \forall w \in W, \quad (3.51)$$

$$(A^{-1} \epsilon_2, \mathbf{v}) - (r_2, \text{div} \mathbf{v}) = -(g_1'(\mathbf{p}^*) - g_1'(\mathbf{p}(u_h^*)), \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}, \quad (3.52)$$

$$(\text{div} \epsilon_2, w) + (r_2 u^*, w) = (g_2'(y^*) - g_2'(y(u_h^*)), w) - (z(u_h^*)(u^* - u_h^*), w), \quad \forall w \in W. \quad (3.53)$$

Se

$$\begin{aligned}\kappa_1(w_h) &= -(y(u_h^*)(u^* - u_h^*), w_h), \\ \kappa_2(v_h) &= -(g_1'(p^*) - g_1'(p(u_h^*)), v_h), \\ \kappa_3(w_h) &= (g_2'(y^*) - g_2'(y(u_h^*)), w_h) - (z(u_h^*)(u^* - u_h^*), w_h).\end{aligned}$$

Since the terms $\kappa_1(w_h)$, $\kappa_2(v_h)$, and $\kappa_3(w_h)$ can be regarded as linear functionals of v_h and w_h defined on V_h and W_h respectively, from the stability results of Ref. [4]

$$\|\epsilon_1\|_{\text{div}} + \|r_1\|_0 \leq C \left\{ \sup_{w_h \in W_h} \frac{|\kappa_1(w_h)|}{\|w_h\|_0} \right\}, \quad (3.54)$$

$$\|\epsilon_2\|_{\text{div}} + \|r_2\|_0 \leq C \left\{ \sup_{v_h \in V_h} \frac{|\kappa_2(v_h)|}{\|v_h\|_{\text{div}}} + \sup_{w_h \in W_h} \frac{|\kappa_3(w_h)|}{\|w_h\|_0} \right\}. \quad (3.55)$$

Hence

$$\begin{aligned}|\kappa_1(w_h)| &= |(y(u_h^*)(u^* - u_h^*), w_h)| \\ &\leq \|y(u_h^*)\|_0 \cdot \|u^* - u_h^*\|_0 \cdot \|w_h\|_0 \\ &\leq C \|u^* - u_h^*\|_0 \cdot \|w_h\|_0,\end{aligned} \quad (3.56)$$

$$\begin{aligned}|\kappa_2(v_h)| &= |(g_1'(p^*) - g_1'(p(u_h^*)), v_h)| \\ &\leq \|g_1'(p^*) - g_1'(p(u_h^*))\|_{\text{div}} \cdot \|v_h\|_{\text{div}} \\ &\leq C \|\epsilon_1\|_{\text{div}} \cdot \|v_h\|_{\text{div}},\end{aligned} \quad (3.57)$$

$$\begin{aligned}|\kappa_3(w_h)| &= |(g_2'(y^*) - g_2'(y(u_h^*)), w_h) - (z(u_h^*)(u^* - u_h^*), w_h)| \\ &\leq \|g_2'(y^*) - g_2'(y(u_h^*))\|_0 \cdot \|w_h\|_0 + \|z(u_h^*)(u^* - u_h^*)\|_0 \cdot \|w_h\|_0 \\ &\leq C \|r_1\|_0 \cdot \|w_h\|_0 + C \|u^* - u_h^*\|_0 \cdot \|w_h\|_0,\end{aligned} \quad (3.58)$$

so from (3.54) and (3.56)

$$\|\epsilon_1\|_{\text{div}} + \|r_1\|_0 \leq C \|u^* - u_h^*\|_0. \quad (3.59)$$

Hence from the estimates (3.55), (3.57)-(3.58) and (3.59) we obtain

$$\|\epsilon_2\|_{\text{div}} + \|r_2\|_0 \leq C \|\epsilon_1\|_{\text{div}} + C \|r_1\|_0 + C \|u^* - u_h^*\|_0 \leq C \|u^* - u_h^*\|_0, \quad (3.60)$$

which completes the proof.

We now combine Lemmas 3.1, 3.3, and 3.5-3.6 with the triangle inequality, to reach the following Theorem.

Theorem 3.1. *Let $(p^*, y^*, q^*, z^*, u^*) \in (V \times W)^2 \times K$ and $(p_h^*, y_h^*, q_h^*, z_h^*, u_h^*) \in (V_h \times W_h)^2 \times K_h$ be the solutions of (2.4)-(2.8) and (2.12)-(2.16), respectively. In addition, assume that $(J_h'(u_h^*))|_T \in H^1(T)$, $\forall T \in \mathcal{T}_h$, and that there is a $v_h \in K_h$ such that*

$$|J_h'(u_h^*), v_h - u^*| \leq C \sum_{T \in \mathcal{T}_h} h_T \|J_h'(u_h^*)\|_{H^1(T)} \cdot \|u^* - u_h^*\|_{L^2(T)}.$$

Then

$$\|\mathbf{p}^* - \mathbf{p}_h^*\|_{\text{div}}^2 + \|y^* - y_h^*\|_0^2 + \|\mathbf{q}^* - \mathbf{q}_h^*\|_{\text{div}}^2 + \|z^* - z_h^*\|_0^2 + \|u^* - u_h^*\|_0^2 \leq C \sum_{i=1}^3 \eta_i^2, \quad (3.61)$$

where η_1 , η_2 , and η_3 are defined in Lemma 3.1, Lemma 3.3 and Lemma 3.5, respectively.

4. An Adaptive Algorithm

In this Section, we introduce an adaptive algorithm to guide the mesh refinement process. *A posteriori* error estimates derived in Section 3 are used as the error indicator, to guide the mesh refinement in the adaptive finite element method.

Let us first discuss the adaptive mesh refinement strategy. The general idea is to refine the mesh such that an error indicator like η is equally distributed over the computational mesh. Assume that an *a posteriori* error estimator η has the form $\eta^2 = \sum_{T_i} \eta_{T_i}^2$, where the T_i denote the finite elements. At each iteration, an average quantity of all $\eta_{T_i}^2$ is calculated, and each $\eta_{T_i}^2$ is then compared with this quantity. The element T_i is to be refined if $\eta_{T_i}^2$ is larger than this quantity. As $\eta_{T_i}^2$ represents the total approximation error over T_i , this strategy ensures that a higher density of nodes is distributed over areas where the error is higher. On this basis, we can define an adaptive algorithm of the optimal control problems (1.1)-(1.4).

Starting from an initial triangulation \mathcal{T}_{h_0} of Ω , we construct a sequence of refined triangulation \mathcal{T}_{h_j} as follows. Given \mathcal{T}_{h_j} , compute the solutions $(\mathbf{p}_h^*, y_h^*, \mathbf{q}_h^*, z_h^*, u_h^*)$ of the system (2.12)-(2.16) and their error estimator

$$\begin{aligned} \eta_c^2 &= h_T^2 \|j'(u_h^*) - y_h^* z_h^*\|_{H^1(T)}, \\ \eta_s^2 &= \|f - \text{div} \mathbf{p}_h^* - y_h^* u_h^*\|_{0,T}^2 + h_T^2 \cdot \|\text{Curl}_h(A^{-1} \mathbf{p}_h^*)\|_{0,T}^2 \\ &\quad + \|h_E^{1/2} [(A^{-1} \mathbf{p}_h^*) \cdot \mathbf{t}]\|_{0,\partial T}^2 + h_T^2 \cdot \|A^{-1} \mathbf{p}_h^*\|_{0,T}^2 \\ &\quad + \|g_2'(y_h^*) - \text{div} \mathbf{q}_h^* - z_h^* u_h^*\|_{0,T}^2 + h_T^2 \cdot \|\text{Curl}_h(A^{-1} \mathbf{q}_h^* + g_1'(\mathbf{p}_h^*))\|_{0,T}^2 \\ &\quad + \|h_E^{1/2} [(A^{-1} \mathbf{q}_h^* + g_1'(\mathbf{p}_h^*)) \cdot \mathbf{t}]\|_{0,\partial T}^2 + h_T^2 \cdot \|A^{-1} \mathbf{q}_h^* + g_1'(\mathbf{p}_h^*)\|_{0,T}^2, \\ E_j &= \sum_{T \in \mathcal{T}_h} (\eta_c^2 + \eta_s^2), \end{aligned}$$

and then adopt the following mesh refinement strategy. Divide all the triangles $T \in \mathcal{T}_{h_j}$ satisfying

$$\eta_T^2 \geq \alpha E_j / n$$

into four new triangles in $\mathcal{T}_{h_{j+1}}$ by joining the midpoints of the edges, where n is the number of the elements of \mathcal{T}_{h_j} , α (a given constant). In order for the new triangulation

$\mathcal{T}_{h_{j+1}}$ to be regular and conformal, it may be necessary to divide some additional triangles into two or four new triangles, depending on whether they have one or more refined neighbours. The new mesh adopted is then $\mathcal{T}_{h_{j+1}}$, and the procedure is continued until $E_j \leq tol$, a given error tolerance.

5. A Numerical Example

We now provide an example to illustrate our theoretical results. The problem considered is

$$\min_{u \in KCU} \frac{1}{2} \{ \|\mathbf{p} - \mathbf{p}_d\|^2 + \|y - y_d\|^2 + \|u - u_d\|^2 \} \quad (5.1)$$

$$\operatorname{div} \mathbf{p} + yu = f, \quad \mathbf{p} = -\nabla y, \quad x \in \Omega, \quad y|_{\partial\Omega} = 0, \quad (5.2)$$

$$\operatorname{div} \mathbf{q} + zu = y - y_d, \quad \mathbf{q} = -\nabla z - \mathbf{p} + \mathbf{p}_d, \quad x \in \Omega, \quad z|_{\partial\Omega} = 0. \quad (5.3)$$

We choose the domain $\Omega = [0, 1] \times [0, 1]$. If $K = \{u \in U : u \geq 0\}$ then $u = \max(u_d + yz, 0)$, and we adopt the iterative scheme

$$b(u_{n+\frac{1}{2}}, v) = b(u_n, v) - \rho(J'(u_n), v), \quad \forall v \in U, \quad (5.4)$$

$$u_{n+1} = P_K(u_{n+\frac{1}{2}}), \quad (5.5)$$

where $b(\cdot, \cdot)$ is a symmetric and positive definite bilinear form and the projection operator P_K can be computed in the same way as in Ref. [17].

We assume that

$$\lambda = \begin{cases} 0.5, & x_1 + x_2 > 1.0, \\ 0.0, & x_1 + x_2 \leq 1.0, \end{cases}$$

and

$$u_d(x_1, x_2) = 1 - \sin \frac{\pi x_1}{2} - \sin \frac{\pi x_2}{2} + \lambda,$$

such that the control function is given by

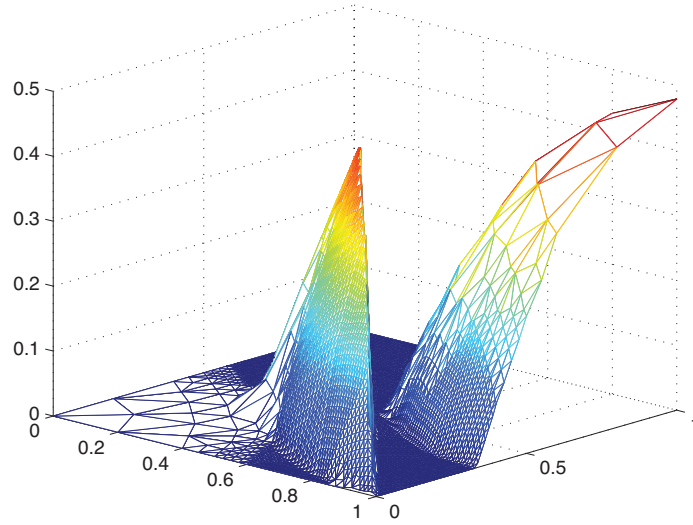
$$u(x_1, x_2) = \max(u_d + yz, 0).$$

We choose the state function by

$$y(x_1, x_2) = \sin 2\pi x_1 \sin 2\pi x_2$$

and the function $f(x_1, x_2) = 8\pi^2 y + uy$ with

$$\mathbf{p}(x_1, x_2) = \mathbf{p}_d(x_1, x_2) = (2\pi \sin 2\pi x_2 \cos 2\pi x_1, 2\pi \sin 2\pi x_1 \cos 2\pi x_2).$$

Figure 1: The profile of the control solution u .

The co-state function can be chosen as

$$z(x_1, x_2) = 2 \sin 2\pi x_1 \sin 2\pi x_2 .$$

It follows from equations (5.2)-(5.3) that

$$\begin{aligned} y_d(x_1, x_2) &= (1 - 16\pi^2)y - uz, \\ \mathbf{q}(x_1, x_2) &= (4\pi \sin 2\pi x_2 \cos 2\pi x_1, 4\pi \sin 2\pi x_1 \cos 2\pi x_2) . \end{aligned}$$

We shall use η_c to denote the control mesh refinement indicator and η_s for the coupled states.

Fig. 1 shows the profile of the control solution u ; and Table 1 lists the error estimates of the control u , the state \mathbf{p} , y , and the co-state \mathbf{q} , z on uniform meshes and on adaptive meshes. For this numerical example, our *a posteriori* error estimators are used as error indicators to guide the mesh refinement in the adaptive finite element method. From Table 1, it can be seen that one may use seven times fewer degrees of freedoms of u on the adaptive meshes to produce a given control error reduction, so the adaptive multi-mesh finite element method is clearly more efficient.

6. Conclusion and Future Work

A posteriori error estimates for mixed finite element approximation of general bilinear optimal control problems have been derived, and more efficient and reliable adaptive finite element methods constructed. Our *a posteriori* error estimates for the numerical solution of the bilinear optimal control problems by mixed finite element methods seem to be new.

Table 1: Numerical results on the uniform mesh and adaptive mesh.

mesh information	uniform mesh (1)	uniform mesh (2)	adaptive mesh (1)	adaptive mesh (2)
u_nodes	7876	32065	1102	1969
u_sides	23282	95552	3143	5744
$u_elements$	15407	63488	2042	3776
yz_nodes	7876	32065	2435	3967
yz_sides	23282	95552	6471	11573
$yz_elements$	15407	63488	4037	7607
$\ u^* - u_h^*\ _{L^2(\Omega)}$	1.41734e-02	8.57621e-03	1.30121e-02	8.27895e-03
$\ y^* - y_h^*\ _{L^2(\Omega)}$	8.18134e-03	3.82328e-03	9.08974e-03	3.56383e-03
$\ z^* - z_h^*\ _{L^2(\Omega)}$	1.63635e-03	7.84240e-04	1.79794e-03	7.03292e-04
$\ p^* - p_h^*\ _{div}$	6.45923e-01	3.23110e-01	7.01679e-01	2.99028e-01
$\ q^* - q_h^*\ _{div}$	1.29847e+0	6.53252e-01	1.44113e+0	6.11254e-01

In future work, we intend to use the mixed finite element method to deal with general non-smooth distributed parameter systems, and to consider *a priori* error estimates and superconvergence for mixed finite element methods of distributed parameter systems.

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