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Identification of a Corroded Boundary and its Robin Coefficient

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Abstract. An inverse geometric problem for two-dimensional Helmholtz-type equations arising in corrosion detection is considered. This problem involves determining an unknown corroded portion of the boundary of a two-dimensional domain and possibly its surface heat transfer (impedance) Robin coefficient from one or two pairs of boundary Cauchy data (boundary temperature and heat flux), and is solved numerically using the meshless method of fundamental solutions. A nonlinear unconstrained minimisation of the objective function is regularised when noise is added into the input boundary data. The stability of the numerical results is investigated for several test examples, with respect to noise in the input data and various values of the regularisation parameters.

AMS subject classifications: 65N20, 65N21, 65N80

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1. Introduction

Inverse geometric problems arise in analysing various imaging and tomography techniques such as electrical impedance tomography (EIT), gamma ray emission tomography (GRET), magneto-resonance imaging (MRI), etc. In this study, we consider the application of the method of fundamental solutions (MFS) to solve numerically the inverse geometric problem, which consists of determining an unknown part of the boundary $\Gamma_2 \subset \partial\Omega$ assuming that the dependent variable u satisfies the Helmholtz (or the modified Helmholtz) equation in a simply-connected bounded domain $\Omega \subset \mathbb{R}^2$ — viz.

$$\nabla^2 u \pm k^2 u = 0 \quad \text{in } \Omega \tag{1.1}$$

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where $k > 0$, from the knowledge of the Dirichlet boundary data $u|_{\Gamma_1}$ and the Neumann flux data $\partial u/\partial n$ — i.e. Cauchy data, on the known part of the boundary $\Gamma_1 = \partial\Omega \setminus \Gamma_2$ where $\underline{n}$ is the outward unit normal to the boundary, together with a boundary condition (Dirichlet, Neumann or Robin) on the unknown part of the boundary Γ_2. Eq. (1.1) with minus sign is the modified Helmholtz equation that models the heat conduction in a fin (e.g. [22]), whilst equation (1.1) with plus sign is the Helmholtz equation that models wave propagation in acoustics. The inverse, nonlinear and ill-posed problem of determining the unknown (inaccessible) corroded portion of the boundary Γ_2 and possibly its surface heat transfer coefficient, if a Robin condition is prescribed on Γ_2, is approached using an MFS regularised minimisation procedure. This study is general and builds upon previous recent applications of the MFS to solve similar boundary determination corrosion problems for the isotropic, anisotropic and functionally graded Laplace equation [20, 24, 26, 27, 32], Helmholtz-type equations [23], the biharmonic equation [33], the Lamé system in elasticity [21], and the heat equation [10]. For more details about the MFS, as applied to inverse problems in general, see the recent review by Karageorghis *et al.* [15]. We finally mention that there also exists an extensive literature on using the boundary element method (BEM) instead of the MFS for the corrosion boundary identification — e.g. see [17] for the Laplace equation in EIT, [25] for the Lamé system in elasticity, and [19] for Helmholtz-type equations. However, there are clear methodological differences between the MFS and the BEM — e.g. see [1] for a comparison between the two methods. In summary, although the MFS formulation may introduce some extra ill-conditioning, by avoiding the numerical integration it is considerably easier to use, especially in higher dimensional problems.

The outline of this paper is as follows. In Section 2 we introduce and discuss the mathematical formulation, whilst in Section 3 we present the MFS for the Helmholtz-type equations. In Section 4 we present and discuss the numerically obtained results, and in Section 5 we give some conclusions and suggest possible future work.

2. Mathematical Formulation

We consider a simply-connected solution domain Ω bounded by a smooth or piecewise smooth curve $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \cap \Gamma_2 = \emptyset$ and both Γ_1 and Γ_2 are of positive measure. The function u satisfies the Helmholtz (or the modified Helmholtz) equation (1.1) subject to the boundary conditions

$$u = f \qquad \text{on} \quad \Gamma_1 \tag{2.1}$$

and

$$\frac{\partial u}{\partial n} + \alpha u = h \qquad \text{on} \quad \Gamma_2\,, \tag{2.2}$$

where $f \in H^{1/2}(\partial\Omega)$ non-constant and $h \in H^{-1/2}(\partial\Omega)$ are given functions, and $\alpha \in L^\infty(\Gamma_2)$ is the non-negative impedance (surface heat transfer) Robin coefficient. Here