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A High-Order Difference Scheme for the Generalized Cattaneo Equation

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Abstract. A high-order finite difference scheme for the fractional Cattaneo equation is investigated. The L_1 approximation is invoked for the time fractional part, and a compact difference scheme is applied to approximate the second-order space derivative. The stability and convergence rate are discussed in the maximum norm by the energy method. Numerical examples are provided to verify the effectiveness and accuracy of the proposed difference scheme.

AMS subject classifications: 65M06, 65M12, 65M15, 35Q51

Key words: Fractional Cattaneo equation, L_1 approximation, compact finite difference, stability, convergence.

1. Introduction

In this paper, we consider the numerical solution of a generalized Cattaneo equation [5,17] with a non-homogeneous term $f(x,t)$:

$$\frac{\partial u(x,t)}{\partial t}+\gamma\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=D\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t)\,, \tag{1.1}$$

where γ is a nonnegative constant related to the relaxation time, D is the diffusion constant, and $f(x,t)$ is a known function. The notation $\partial^\alpha/\partial t^\alpha$ in (1.1) denotes the time fractional derivative operator based on Caputo's definition [9,16], given by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\equiv\frac{1}{\Gamma(2-\alpha)}\int_0^t\frac{\partial^2 u(x,s)}{\partial s^2}\frac{ds}{(t-s)^{\alpha-1}}\,,\qquad \alpha\in(1,2)\,, \tag{1.2}$$

where $\Gamma(\cdot)$ is the gamma function. The standard Cattaneo equation is normally obtained by using a generalized form of Fick's law [8]. The equation describes a diffusion process

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with a finite velocity of propagation and has a variety of applications — e.g. extended irreversible thermodynamics [8], modelling both heat and mass transfer [7], (inflationary) cosmological models [27], and the diffusion theory in crystalline solids [6]. However, the classical Cattaneo equation cannot describe the anomalous diffusion behavior observed in many natural systems. To address this issue, Compte and Metzler [1] generalized the classical Cattaneo model to the time fractional Cattaneo models and studied the properties of the corresponding fractional Cattaneo equations in both the long-time and short-time limits. Following Compte and Metzler, Kosztołowicz and Lewandowska [10] presented a theoretical foundation for studies of the subdiffusion impedance using a hyperbolic equation. Povstenko [17] considered the generalized Cattaneo-type equations with time fractional derivatives and formulated the corresponding theory of thermal stresses. Qi and Jiang [18] extended the classical Cattaneo equation to the space-time fractional Cattaneo equation and derived the exact solution by joint Laplace and Fourier transforms.

Although some theoretical analysis has been presented for the generalized Cattaneo equations [1, 17, 18], little work has been done on numerical methods. Currently, Ghazizadeh *et al.* [5] derive the generalized Cattaneo equation using a concept of single-phase lag equation [25] and a recently introduced fractional Taylor series expansion formula [15]. Two finite difference schemes, namely an explicit predictor-corrector scheme and a totally implicit scheme, have been developed [5]. In recent years, some numerical methods have been proposed for solving other types of fractional differential equations. Meerschaert and Tadjeran [13,14] investigated space-fractional differential equations, and proposed an implicit Euler method based on a shifted Grünwald formula to approximate fractional derivatives of order $1 < \alpha < 2$. Yuste and Acedo [26] proposed an explicit finite difference method and analyzed the stability condition for the fractional subdiffusion equation. Langlands and Henry [11] also considered this type of equation, and constructed an implicit finite difference by using the L_1 scheme to approximate the fractional derivative. The accuracy and stability were discussed by the Fourier method. Zhuang *et al.* [28] studied the stability and convergence of an implicit numerical method by the energy method. Cui [2] used a fourth-order compact difference scheme to increase the spatial accuracy for solving the fractional anomalous subdiffusion equation with a nonhomogeneous term. Du *et al.* [3] derived a compact difference scheme for the fractional diffusion-wave equation based on the L_1 approximation. Gao and Sun [4] first transformed the original fractional subdiffusion problem to an equivalent form and then applied the compact difference scheme with the L_1 approximation to discretize the resulting equation. By introducing a new inner product, they analyzed the stability and convergence of the proposed scheme by the energy method. For relevant main elements and ideas, reference can be made to the original papers in Refs. [12, 23].

We consider the numerical solution of the generalized Cattaneo equation (1.1). In Section 2, two new variables are introduced to transform the original equation (1.1) into a low order system of equations (cf. [21]), and the numerical solution of the low order equation is then investigated by applying the L_1 approximation to the time fractional derivative and the compact difference scheme to the second-order space derivative (cf. [3,4]). Theoretical analysis in Section 3 shows that the resulting difference scheme is unconditionally stable and the convergence order is $\mathcal{O}(h^4+\tau^{3-\alpha})$ in the maximum norm, where h is the space step and τ is the time step. Numerical experiments discussed in Section 4 demonstrate our theoretical results, and concluding remarks are made in Section 5.

2. Finite Difference Scheme

For simplicity, we take the parameters in (1.1) to be unity and consider the equation

$$\frac{\partial u(x,t)}{\partial t}+\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t)\,,\qquad a\le x\le b\,,\quad 0<t\le T, \tag{2.1}$$

subject to the initial conditions

$$u(x,0)=\phi(x)\,,\qquad \frac{\partial u(x,0)}{\partial t}=\psi(x)\,,\qquad a\le x\le b\,, \tag{2.2}$$

and the boundary conditions

$$u(a,t)=\varphi_L(t)\,,\qquad u(b,t)=\varphi_R(t)\,,\qquad 0<t\le T\,, \tag{2.3}$$

where $\phi(x)$, $\psi(x)$, $\varphi_L(t)$, $\varphi_R(t)$ are known functions and $\alpha\in(1,2)$. Using the technique proposed in Refs , [3,4], we now develop our convergent finite difference scheme.

Let $h=(b-a)/M$, $\tau=T/N$, $x_j=a+jh$, $t_k=k\tau$, $u(x_j,t_k)=u_j^k$, $u^k=(u_0^k,u_1^k,\cdots,u_M^k)$ for $j=0,1,\cdots,M$ and $k=0,1,\cdots,N$, where M and N are certain positive integers. We also introduce

$$\delta_x u^k_{j-\frac12}=\frac1h\left(u^k_j-u^k_{j-1}\right)\,,\qquad \delta_x^2 u^k_j=\frac1h\left(\delta_x u^k_{j+\frac12}-\delta_x u^k_{j-\frac12}\right)\,, \tag{2.4}$$

$$\mathscr{H}_h[u_j^k]=\begin{cases}\left(1+\dfrac{h^2}{12}\delta_x^2\right)u_j^k=\dfrac{1}{12}\left(u^k_{j-1}+10u^k_j+u^k_{j+1}\right)\,, & 1\le j\le M-1\,,\\[2ex] u_j^k\,, & j=0,M\,.\end{cases} \tag{2.5}$$

With this notation, we define the inner product and norms — viz.

$$\langle u^k,v^k\rangle\equiv h\sum_{j=0}^{M}u_j^kv_j^k-\frac{h^3}{12}\sum_{j=0}^{M-1}\left(\delta_x u^k_{j+\frac12}\right)\left(\delta_x v^k_{j+\frac12}\right)\,, \tag{2.6}$$

and

$$\|u^k\|_\infty\equiv\max_{0\le j\le M}|u_j^k|\,,\qquad |u^k|_1\equiv\left[h\sum_{j=0}^{M-1}\left(\delta_x u^k_{j+\frac12}\right)^2\right]^{1/2},\qquad \|u^k\|\equiv\left[h\sum_{j=0}^{M}(u_j^k)^2\right]^{1/2}. \tag{2.7}$$

Note that

$$h^2|u^k|_1^2=h^3\sum_{j=0}^{M-1}\left(\delta_x u^k_{j+\frac12}\right)^2=h\sum_{j=0}^{M-1}(u^k_{j+1}-u^k_j)^2\le 4\|u^k\|^2\,,$$

and from Eqs. (2.6) and (2.7) we get

$$\frac{2}{3}\|u^k\|^2 \leq \langle u^k, u^k\rangle \leq \|u^k\|^2 . \tag{2.8}$$

For $1 \leq j \leq M-1$ and $1 \leq k \leq N$, the difference scheme considered for (2.1)–(2.3) is

$$\begin{aligned}&\mathscr{H}_h[\delta_t U_j^{k-\frac{1}{2}}] + \frac{1}{\tau\Gamma(2-\alpha)}\mathscr{H}_h\left[a_0\delta_t U_j^{k-\frac{1}{2}} - \sum_{l=1}^{k-1}(a_{k-l-1}-a_{k-l})\delta_t U_j^{l-\frac{1}{2}} - a_{k-1}\psi_j\right]\\ &= \delta_x^2 U_j^{k-\frac{1}{2}} + \mathscr{H}_h[f_j^{k-\frac{1}{2}}]\end{aligned} \tag{2.9}$$

on noting that when $k=1$ the summation term vanishes, with

$$U_j^0 = \phi_j , \qquad U_0^k = \varphi_L^k , \qquad U_M^k = \varphi_R^k , \tag{2.10}$$

where $\phi_j = \phi(x_j)$, $\psi_j = \psi(x_j)$ and $\varphi_L^k = \varphi_L(t_k)$, $\varphi_R^k = \varphi_R(t_k)$, $f_j^k = f(x_j, t_k)$ and

$$a_l = \int_{t_l}^{t_{l+1}} \frac{\mathrm{d}t}{t^{\alpha-1}} = \frac{\tau^{2-\alpha}}{2-\alpha}\left[(l+1)^{2-\alpha} - l^{2-\alpha}\right] . \tag{2.11}$$

Here we denote $\delta_x^2 U_j^{k-1/2} = (\delta_x^2 U_j^k + \delta_x^2 U_j^{k-1})/2$, and use similar notation for other intermediate values of the grid functions throughout the paper.

To derive the above difference scheme (2.9)–(2.11), we invoke the following lemmas.

Lemma 2.1 ([21]). *Suppose* $g(t) \in \mathbf{C}^2[0, t_k]$. *Then*

$$\begin{aligned}&\left|\int_0^{t_k} \frac{g'(t)}{(t_k-t)^{\alpha-1}}\mathrm{d}t - \frac{1}{\tau}\left[a_0 g(t_k) - \sum_{l=1}^{k-1}(a_{k-l-1}-a_{k-l})g(t_l) - a_{k-1}g(t_0)\right]\right|\\ \leq &\frac{1}{2-\alpha}\left[\frac{2-\alpha}{12} + \frac{2^{3-\alpha}}{3-\alpha} - (1+2^{1-\alpha})\right]\max_{0\leq t\leq t_k}|g''(t)|\tau^{3-\alpha} ,\end{aligned}$$

where a_l *is defined in* (2.11) *and* $1 < \alpha < 2$.

Lemma 2.2 ([4,20]). *Suppose* $f(x) \in \mathbf{C}^6[a,b]$ *and* $p(\xi) = 5(1-\xi)^3 - 3(1-\xi)^5$. *Then*

$$\mathscr{H}_h f''(x_j) = \delta_x^2 f(x_j) + \frac{h^4}{360}\int_0^1 [f^{(6)}(x_j - \xi h) + f^{(6)}(x_j + \xi h)]p(\xi)\mathrm{d}\xi , \qquad 1 \leq j \leq M-1 .$$

Let $u(x,t)$ be the exact solution of (2.1)–(2.3) and define

$$v(x,t) = \frac{\partial u(x,t)}{\partial t} , \qquad w(x,t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{\partial v(x,s)}{\partial s}\frac{ds}{(t-s)^{\alpha-1}} ,$$

so Eq. (2.1) can be rewritten as

$$v(x,t)+w(x,t)=\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t)\,. \tag{2.12}$$

Denote $v_j^k=v(x_j,t_k)$ and $w_j^k=w(x_j,t_k)$. Using the Taylor expansion, one can easily see that

$$v_j^{k-\frac{1}{2}}=\delta_t u_j^{k-\frac{1}{2}}+(r_1)_j^{k-\frac{1}{2}}\,, \tag{2.13}$$

with $\left|(r_1)_j^{k-1/2}\right|\le c_1\tau^2$ for some constant c_1. From Lemma 2.1, we also have

$$\begin{aligned} w_j^{k-\frac{1}{2}}=&\frac{1}{2}(w_j^k+w_j^{k-1})\\ =&\frac{1}{\tau\Gamma(2-\alpha)}\left[a_0 v_j^{k-\frac{1}{2}}-\sum_{l=1}^{k-1}(a_{k-l-1}-a_{k-l})v_j^{l-\frac{1}{2}}-a_{k-1}v_j^0\right]+(r_2)_j^{k-\frac{1}{2}}\,, \end{aligned} \tag{2.14}$$

with $|(r_2)_j^{k-1/2}|\le c_2\tau^{3-\alpha}$ for some constant c_2. Substituting Eqs. (2.13) and (2.14) into Eq. (2.12) and noting that $v_j^0=v(x_j,0)=\psi(x_j)$, we obtain

$$\begin{aligned} &\delta_t u_j^{k-\frac{1}{2}}+\frac{1}{\tau\Gamma(2-\alpha)}\left[a_0\delta_t u_j^{k-\frac{1}{2}}-\sum_{l=1}^{k-1}(a_{k-l-1}-a_{k-l})\delta_t u_j^{l-\frac{1}{2}}-a_{k-1}\psi_j\right]\\ =&\frac{\partial u_j^{k-\frac{1}{2}}}{\partial x^2}+f_j^{k-\frac{1}{2}}+(r_3)_j^{k-\frac{1}{2}}\,, \end{aligned}$$

with $|(r_3)_j^{k-1/2}|\le c_3\tau^{3-\alpha}$ for some constant c_3. Applying the operator $\mathscr{H}_h$ defined by (2.5) to both sides of this equation, from Lemma 2.2 we get

$$\begin{aligned} &\mathscr{H}_h[\delta_t u_j^{k-\frac{1}{2}}]+\frac{1}{\tau\Gamma(2-\alpha)}\mathscr{H}_h\left[a_0\delta_t u_j^{k-\frac{1}{2}}-\sum_{l=1}^{k-1}(a_{k-l-1}-a_{k-l})\delta_t u_j^{l-\frac{1}{2}}-a_{k-1}\psi_j\right]\\ =&\,\delta_x^2 u_j^{k-\frac{1}{2}}+\mathscr{H}_h[f_j^{k-\frac{1}{2}}]+R_j^{k-\frac{1}{2}}, \end{aligned} \tag{2.15}$$

where

$$R_j^{k-\frac{1}{2}}=\frac{h^4}{360}\int_0^1\left[\frac{\partial^6}{\partial x^6}u\left(x_j-\xi h,t_{k-\frac{1}{2}}\right)+\frac{\partial^6}{\partial x^6}u\left(x_j+\xi h,t_{k-\frac{1}{2}}\right)\right]p(\xi)\mathrm{d}\xi+\mathscr{H}_h\left[(r_3)_j^{k-\frac{1}{2}}\right]$$

and there exists a constant c_4 such that

$$\left|R_j^{k-\frac{1}{2}}\right|\le c_4(h^4+\tau^{3-\alpha})\,. \tag{2.16}$$

In addition, Eqs. (2.2) and (2.3) yield

$$u_j^0=\phi(x_j),\ u_0^k=\varphi_L(t_k)\,,\ u_M^k=\varphi_R(t_k)\,,\ 0\le j\le M\,,\ 1\le k\le N\,. \tag{2.17}$$

On omitting the small term $R_j^{k-1/2}$, we thereby have the difference scheme (2.9)–(2.11) for (2.1)–(2.3).

3. Stability and Convergence Analysis

The stability and convergence of the scheme (2.9)–(2.11) is investigated using the following lemmas.

Lemma 3.1 ([21]). *Let a_k be defined in* (2.11)*, $k=0,1,2,\cdots$. Then*

(i) $\dfrac{\tau^{2-\alpha}}{2-\alpha}=a_1>a_2>\cdots>a_k\to 0$*, as $k\to+\infty$;*

(ii) $\displaystyle\sum_{k=1}^{n}a_{k-1}=\frac{t_n^{2-\alpha}}{2-\alpha}$.

Lemma 3.2 ([19]). *Let $v=(v_0,v_1,\cdots,v_M)$ with $v_0=v_M=0$. Then $\|v\|_\infty\le\sqrt{b-a}\,|v|_1/2$.*

Lemma 3.3. *Suppose $\{u_j^k\}$ is the solution of*

$$\mathscr{H}_h\left[\delta_t u_j^{k-\frac{1}{2}}\right]+\frac{1}{\tau\Gamma(2-\alpha)}\mathscr{H}_h\left[a_0\delta_t u_j^{k-\frac{1}{2}}-\sum_{l=1}^{k-1}(a_{k-l-1}-a_{k-l})\delta_t u_j^{l-\frac{1}{2}}-a_{k-1}q_j\right]$$
$$=\delta_x^2 u_j^{k-\frac{1}{2}}+g_j^{k-\frac{1}{2}}\,, \tag{3.1}$$

and

$$u_j^0=\phi_j\,,\qquad u_0^k=u_M^k=0\,,\qquad 1\le j\le M-1\,,\quad 1\le k\le N\,. \tag{3.2}$$

Then

$$|u^k|_1^2\le|u^0|_1^2+\frac{t_k^{2-\alpha}}{\Gamma(3-\alpha)}\langle q,q\rangle+\frac{3}{4}\tau\sum_{l=1}^{k}h\sum_{j=1}^{M-1}(g_j^{l-\frac{1}{2}})^2,\qquad 1\le k\le N\,.$$

Proof. From Eq. (3.2), $\delta_t u_0^{k-1/2}=\delta_t u_M^{k-1/2}=0$. Then for any v_j $(0\le j\le M)$ we have from (2.4) that

$$\begin{aligned}\sum_{j=1}^{M-1}\delta_t u_j^{k-\frac{1}{2}}\delta_x^2 v_j=&\frac{1}{h}\sum_{j=1}^{M-1}\delta_t u_j^{k-\frac{1}{2}}\left(\delta_x v_{j+\frac{1}{2}}-\delta_x v_{j-\frac{1}{2}}\right)\\&=-\frac{1}{h}\sum_{j=0}^{M-1}\left(\delta_t u_{j+1}^{k-\frac{1}{2}}-\delta_t u_j^{k-\frac{1}{2}}\right)\delta_x v_{j+\frac{1}{2}}\\&=-\sum_{j=0}^{M-1}\left(\delta_x\delta_t u_{j+\frac{1}{2}}^{k-\frac{1}{2}}\right)\left(\delta_x v_{j+\frac{1}{2}}\right)\,,\end{aligned} \tag{3.3}$$

therefore

$$
\begin{aligned}
\tau h \sum_{k=1}^{n} \sum_{j=1}^{M-1} \delta_t u_j^{k-\frac{1}{2}} \cdot \mathscr{H}_h[v_j] =& \tau h \sum_{k=1}^{n} \sum_{j=1}^{M-1} \delta_t u_j^{k-\frac{1}{2}} \left(1 + \frac{h^2}{12} \delta_x^2\right) v_j \\
=& \tau \sum_{k=1}^{n} \left(h \sum_{j=1}^{M-1} \delta_t u_j^{k-\frac{1}{2}} v_j + \frac{h^3}{12} \sum_{j=1}^{M-1} \delta_t u_j^{k-\frac{1}{2}} \delta_x^2 v_j \right) \\
=& \tau \sum_{k=1}^{n} \left[h \sum_{j=0}^{M} \delta_t u_j^{k-\frac{1}{2}} v_j - \frac{h^3}{12} \sum_{j=0}^{M-1} \left(\delta_x \delta_t u_{j+\frac{1}{2}}^{k-\frac{1}{2}} \right) \left(\delta_x v_{j+\frac{1}{2}} \right) \right] \\
=& \tau \sum_{k=1}^{n} \langle \delta_t u^{k-\frac{1}{2}} v \rangle \,.
\end{aligned}
$$

Then multiplying Eq. (3.1) with $\tau h \delta_t u_j^{k-1/2}$, and summing for j from 1 to $M-1$ and for k from 1 to n where $1 \leq n \leq N$, we obtain

$$
\begin{aligned}
& \tau \sum_{k=1}^{n} \langle \delta_t u^{k-\frac{1}{2}}, \delta_t u^{k-\frac{1}{2}} \rangle + \frac{a_0}{\Gamma(2-\alpha)} \sum_{k=1}^{n} \langle \delta_t u^{k-\frac{1}{2}}, \delta_t u^{k-\frac{1}{2}} \rangle \\
& - \frac{1}{\Gamma(2-\alpha)} \sum_{k=2}^{n} \sum_{l=1}^{k-1} (a_{k-l-1} - a_{k-l}) \langle \delta_t u^{k-\frac{1}{2}}, \delta_t u^{l-\frac{1}{2}} \rangle - \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{n} a_{k-1} \langle \delta_t u^{k-\frac{1}{2}}, q \rangle \\
=& \tau h \sum_{k=1}^{n} \sum_{j=1}^{M-1} \delta_t u_j^{k-\frac{1}{2}} \delta_x^2 u_j^{k-\frac{1}{2}} + \tau h \sum_{k=1}^{n} \sum_{j=1}^{M-1} \delta_t u_j^{k-\frac{1}{2}} g_j^{k-\frac{1}{2}}. \qquad (3.4)
\end{aligned}
$$

We note that

$$
\begin{aligned}
& - \sum_{k=2}^{n} \sum_{l=1}^{k-1} (a_{k-l-1} - a_{k-l}) \langle \delta_t u^{k-\frac{1}{2}}, \delta_t u^{l-\frac{1}{2}} \rangle \\
\geq & -\frac{1}{2} \sum_{k=2}^{n} \sum_{l=1}^{k-1} (a_{k-l-1} - a_{k-l}) \big(\langle \delta_t u^{k-\frac{1}{2}}, \delta_t u^{k-\frac{1}{2}} \rangle + \langle \delta_t u^{l-\frac{1}{2}}, \delta_t u^{l-\frac{1}{2}} \rangle \big) \\
= & -\frac{1}{2} \sum_{k=2}^{n} (a_0 - a_{k-1}) \langle \delta_t u^{k-\frac{1}{2}}, \delta_t u^{k-\frac{1}{2}} \rangle - \frac{1}{2} \sum_{l=1}^{n-1} (a_0 - a_{n-l}) \langle \delta_t u^{l-\frac{1}{2}}, \delta_t u^{l-\frac{1}{2}} \rangle \\
= & -\frac{1}{2} \sum_{k=1}^{n} (2a_0 - a_{k-1} - a_{n-k}) \langle \delta_t u^{k-\frac{1}{2}}, \delta_t u^{k-\frac{1}{2}} \rangle \,, \qquad (3.5)
\end{aligned}
$$

on exchanging the summation order to obtain the first equality, and in addition

$$
- \sum_{k=1}^{n} a_{k-1} \langle \delta_t u^{k-\frac{1}{2}}, q \rangle \geq -\frac{1}{2} \sum_{k=1}^{n} a_{k-1} \left(\langle \delta_t u^{k-\frac{1}{2}}, \delta_t u^{k-\frac{1}{2}} \rangle + \langle q, q \rangle \right) . \qquad (3.6)
$$

Invoking the inequalities (3.5) and (3.6) on the left-hand side of Eq. (3.4), and then also using Lemma 3.1 and inequality (2.8), we have

$$\begin{aligned}
&\tau\sum_{k=1}^{n}\langle\delta_t u^{k-\frac{1}{2}},\delta_t u^{k-\frac{1}{2}}\rangle+\frac{a_0}{\Gamma(2-\alpha)}\sum_{k=1}^{n}\langle\delta_t u^{k-\frac{1}{2}},\delta_t u^{k-\frac{1}{2}}\rangle\\
&-\frac{1}{\Gamma(2-\alpha)}\sum_{k=2}^{n}\sum_{l=1}^{k-1}(a_{k-l-1}-a_{k-l})\langle\delta_t u^{k-\frac{1}{2}},\delta_t u^{l-\frac{1}{2}}\rangle-\frac{1}{\Gamma(2-\alpha)}\sum_{k=1}^{n}a_{k-1}\langle\delta_t u^{k-\frac{1}{2}},q\rangle\\
\geq&\tau\sum_{k=1}^{n}\langle\delta_t u^{k-\frac{1}{2}},\delta_t u^{k-\frac{1}{2}}\rangle+\frac{1}{2\Gamma(2-\alpha)}\sum_{k=1}^{n}a_{n-k}\langle\delta_t u^{k-\frac{1}{2}},\delta_t u^{k-\frac{1}{2}}\rangle-\frac{1}{2\Gamma(2-\alpha)}\sum_{k=1}^{n}a_{k-1}\langle q,q\rangle\\
\geq&\tau\sum_{k=1}^{n}\langle\delta_t u^{k-\frac{1}{2}},\delta_t u^{k-\frac{1}{2}}\rangle-\frac{t_n^{2-\alpha}}{2(2-\alpha)\Gamma(2-\alpha)}\langle q,q\rangle\\
\geq&\frac{2}{3}\tau\sum_{k=1}^{n}\|\delta_t u^{k-\frac{1}{2}}\|^2-\frac{t_n^{2-\alpha}}{2\Gamma(3-\alpha)}\langle q,q\rangle\,.
\end{aligned}\tag{3.7}$$

For the right-hand side of (3.4), from (3.3) we have

$$\begin{aligned}
\tau h\sum_{k=1}^{n}\sum_{j=1}^{M-1}\delta_t u_j^{k-\frac{1}{2}}\delta_x^2 u_j^{k-\frac{1}{2}}=&-\tau\sum_{k=1}^{n}\left[h\sum_{j=0}^{M-1}(\delta_x\delta_t u_{j+\frac{1}{2}}^{k-\frac{1}{2}})(\delta_x u_{j+\frac{1}{2}}^{k-\frac{1}{2}})\right]\\
=&-\frac{1}{2}\sum_{k=1}^{n}\left[h\sum_{j=0}^{M-1}(\delta_x u_{j+\frac{1}{2}}^{k})^2-h\sum_{j=0}^{M-1}(\delta_x u_{j+\frac{1}{2}}^{k-1})^2\right]\\
=&-\frac{1}{2}\left(|u^n|_1^2-|u^0|_1^2\right)\,.
\end{aligned}\tag{3.8}$$

Moreover,

$$\begin{aligned}
\tau h\sum_{k=1}^{n}\sum_{j=1}^{M-1}\delta_t u_j^{k-\frac{1}{2}}g_j^{k-\frac{1}{2}}\leq&\frac{1}{2}\tau\sum_{k=1}^{n}\left[\frac{4}{3}h\sum_{j=1}^{M-1}\left(\delta_t u_j^{k-\frac{1}{2}}\right)^2+\frac{3}{4}h\sum_{j=1}^{M-1}\left(g_j^{k-\frac{1}{2}}\right)^2\right]\\
=&\frac{2}{3}\tau\sum_{k=1}^{n}\|\delta_t u^{k-\frac{1}{2}}\|^2+\frac{3}{8}\tau\sum_{k=1}^{n}h\sum_{j=1}^{M-1}\left(g_j^{k-\frac{1}{2}}\right)^2.
\end{aligned}\tag{3.9}$$

Thus invoking (3.7)–(3.9), from Eq. (3.4) we obtain

$$|u^n|_1^2\leq|u^0|_1^2+\frac{t_n^{2-\alpha}}{\Gamma(3-\alpha)}\langle q,q\rangle+\frac{3}{4}\tau\sum_{k=1}^{n}h\sum_{j=1}^{M-1}\left(g_j^{k-\frac{1}{2}}\right)^2,$$

or

$$|u^k|_1^2\leq|u^0|_1^2+\frac{t_k^{2-\alpha}}{\Gamma(3-\alpha)}\langle q,q\rangle+\frac{3}{4}\tau\sum_{l=1}^{k}h\sum_{j=1}^{M-1}\left(g_j^{l-\frac{1}{2}}\right)^2.$$

□

Theorem 3.1. *The difference scheme* (2.9)–(2.11) *is unconditionally stable for any initial values.*

Proof. Assume $\{v_i^k\}$ is the solution of

$$\mathcal{H}_h\left[\delta_t v_j^{k-\frac{1}{2}}\right] + \frac{1}{\tau\Gamma(2-\alpha)}\mathcal{H}_h\left[a_0\delta_t v_j^{k-\frac{1}{2}} - \sum_{l=1}^{k-1}(a_{k-l-1}-a_{k-l})\delta_t v_j^{l-\frac{1}{2}} - a_{k-1}(\psi_j+\varrho_j)\right]$$
$$= \delta_x^2 v_j^{k-\frac{1}{2}} + \mathcal{H}_h[f_j^{k-\frac{1}{2}}]\,, \tag{3.10}$$

and

$$v_j^0 = \phi_j + \rho_j, \qquad v_0^k = \varphi_L^k, \qquad v_M^k = \varphi_R^k, \qquad 1 \le j \le M-1, \quad 1 \le k \le N\,,$$

where $\varrho = (\varrho_1, \varrho_2, \cdots, \varrho_{M-1})$ and $\rho = (\rho_1, \rho_2, \cdots, \rho_{M-1})$ are perturbation error vectors of the initial data. For $1 \le k \le N$, let

$$\varepsilon^k = (\varepsilon_1^k, \varepsilon_2^k, \ldots, \varepsilon_{M-1}^k)\,,$$

where $\varepsilon_j^k = v_j^k - U_j^k$. Subtracting (2.9)–(2.10) from the above scheme, we obtain the perturbation error equation

$$\mathcal{H}_h\left[\delta_t \varepsilon_j^{k-\frac{1}{2}}\right] + \frac{1}{\tau\Gamma(2-\alpha)}\mathcal{H}_h\left[a_0\delta_t \varepsilon_j^{k-\frac{1}{2}} - \sum_{l=1}^{k-1}(a_{k-l-1}-a_{k-l})\delta_t \varepsilon_j^{l-\frac{1}{2}} - a_{k-1}\varrho_j\right]$$
$$= \delta_x^2 \varepsilon_j^{k-\frac{1}{2}},$$

and

$$\varepsilon_j^0 = \rho_j, \qquad \varepsilon_0^k = 0, \qquad \varepsilon_M^k = 0, \qquad 1 \le j \le M-1, \quad 1 \le k \le N.$$

Applying Lemma 3.3 to this perturbation error equation, we have

$$|\varepsilon^k|_1^2 \le |\varepsilon^0|_1^2 + \frac{t_k^{2-\alpha}}{\Gamma(3-\alpha)}\langle\varrho,\varrho\rangle \le |\varepsilon^0|_1^2 + \frac{T^{2-\alpha}}{\Gamma(3-\alpha)}\langle\varrho,\varrho\rangle\,, \qquad 1 \le k \le N\,,$$

where $\varepsilon^0 = (\varepsilon_1^0, \varepsilon_2^0, \cdots, \varepsilon_{M-1}^0) = (\rho_1, \rho_2, \cdots, \rho_{M-1}) = \rho$. In addition, from Lemma 3.2

$$\|\varepsilon^k\|_\infty \le \left[\frac{b-a}{4}|\rho|_1^2 + \frac{T^{2-\alpha}(b-a)}{4\Gamma(3-\alpha)}\langle\varrho,\varrho\rangle\right]^{1/2}, \qquad 1 \le k \le N\,.$$

so the difference scheme (2.9)–(2.11) is unconditionally stable for any initial values in the maximum norm. □

Theorem 3.2. *Assume that* $u(x,t) \in \mathbf{C}_{x,t}^{6,3}([a,b]\times[0,T])$ *is the solution of the problem* (2.1)–(2.3)*, and* $\{U_j^k \mid 0 \le j \le M, k \ge 0\}$ *is the solution of* (2.9)–(2.11)*. Then for* $1 \le k \le N$ *we have*

$$\|u^k - U^k\|_\infty \le \frac{\sqrt{3T}(b-a)c_4}{4}(h^4 + \tau^{3-\alpha})\,.$$

Proof. From (2.15)–(2.17) and (2.9)–(2.11), the error $e_j^k = U_j^k - u_j^k$ satisfies

$$\mathscr{H}_h\left[\delta_t e_j^{k-\frac{1}{2}}\right] + \frac{1}{\tau\Gamma(2-\alpha)}\mathscr{H}_h\left[a_0\delta_t e_j^{k-\frac{1}{2}} - \sum_{l=1}^{k-1}(a_{k-l-1}-a_{k-l})\delta_t e_j^{l-\frac{1}{2}}\right] = \delta_x^2 e_j^{k-\frac{1}{2}} + R_j^{k-\frac{1}{2}},$$

subject to the conditions

$$e_j^0 = 0\,, \qquad e_0^k = e_M^k = 0\,, \qquad 1 \le j \le M-1\,, \quad 1 \le k \le N\,.$$

From Lemma 3.3 we have

$$|e^k|_1^2 \le \frac{3}{4}\tau\sum_{l=1}^{k} h\sum_{j=1}^{M-1}\big(R_j^{l-\frac{1}{2}}\big)^2\,, \qquad 1 \le k \le N\,;$$

and using inequality (2.16),

$$|e^k|_1 \le \frac{c_4\sqrt{3t_k(b-a)}}{2}(h^4+\tau^{3-\alpha}), \qquad 1 \le k \le N\,.$$

Thus from Lemma 3.2 we obtain

$$\|e^k\|_\infty \le \frac{\sqrt{3t_k}(b-a)c_4}{4}(h^4+\tau^{3-\alpha}) \le \frac{\sqrt{3T}(b-a)c_4}{4}(h^4+\tau^{3-\alpha}), \qquad 1 \le k \le N\,,$$

so the convergence of the difference scheme (2.9)–(2.11) is $\mathscr{O}(h^4+\tau^{3-\alpha})$ in the maximum norm. □

4. Numerical Experiments

Numerical examples illustrate the effectiveness and the accuracy of the proposed difference scheme. In all tables, "M" and "N" are the number of spatial grid points and the number of time steps respectively, "$E_\infty(h,\tau)$" refers to the maximum error of the solution at the last time step (i.e. $E_\infty(h,\tau) = \|u^N - U^N\|_\infty$), and "order" means the convergence order of the difference scheme, respectively equal to $\log_2[E_\infty(h,\tau)/E_\infty(h,\tau/2)]$ for the time direction and $\log_2[E_\infty(h,\tau)/E_\infty(h/2,\tau)]$ for the spatial direction.

Example 4.1. (Homogeneous PDE)

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2}\,, \qquad 0 < x < 1\,, \quad 0 < t \le 1\,, \tag{4.1}$$

$$u(x,0) = 0\,, \qquad \frac{\partial u(x,0)}{\partial t} = \sin(\pi x)\,, \qquad 0 \le x \le 1\,, \tag{4.2}$$

$$u(0,t) = 0\,, \qquad u(1,t) = 0\,, \qquad 0 < t \le 1\,. \tag{4.3}$$

Similar to Ref. [5], we find the exact solution through the Laplace transform and the Riemann sum approximation for the Laplace inversion. Taking the Laplace transform of Eq. (4.1) and accounting for the initial conditions (4.2), we have

$$s\cdot\overline{u}(x,s)+s^{\alpha}\cdot\overline{u}(x,s)-s^{\alpha-2}\cdot\sin(\pi x)=\frac{\mathrm{d}^2\overline{u}(x,s)}{\mathrm{d}x^2}\,, \tag{4.4}$$

where $\overline{u}(x,s)$ denotes the Laplace transform of $u(x,t)$. Correspondingly, boundary conditions (4.3) are transformed to

$$\overline{u}(0,s)=\overline{u}(1,s)=0\,, \tag{4.5}$$

and the solurion to (4.4) and (4.5) is

$$\overline{u}(x,s)=\frac{s^{\alpha-2}}{s+s^{\alpha}+\pi^2}\sin(\pi x)\,. \tag{4.6}$$

Next, we employ the Riemann sum approximation for the Laplace inversion in Eq. (4.6) to obtain

$$u(x,t)=\frac{e^{\zeta t}}{t}\left[\frac{1}{2}\overline{u}(x,\zeta)+\mathrm{Re}\sum_{j=1}^{p}\overline{u}\Big(x,\zeta+\frac{\mathbf{i}j\pi}{t}\Big)(-1)^j\right]\,,$$

where $\zeta=(2+e)/t$, “Re" is the real part of the summation and $\mathbf{i}\equiv\sqrt{-1}$. We adopt $p=10000$, which is adequate for the required accuracy. The exact solutions at time $t=1$ for different values of α are plotted in Fig. 1.

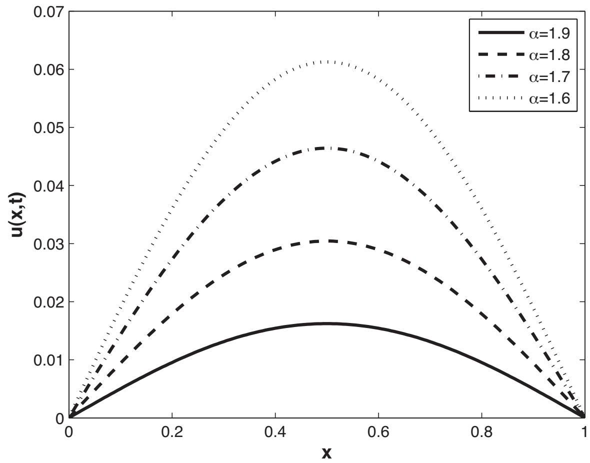

Figure 1: Exact solutions for Example 1 at $t=1$, with different values of α.

Table 1 gives the maximum errors of the numerical solution at time $t=1$, and the convergence order in the time direction when $h=1/10^4$. For different values of α, we see that the convergence order in the time direction is always $\mathcal{O}(\tau^{3-\alpha})$, consistent with our theoretical analysis. To test the convergence order in the spatial direction, we fix the time step size small enough, say $\tau = 1/(4\cdot 10^5)$, and increase the number of spatial grid points. Table 2 lists the numerical results at time $t = 1$. Evidently, the numerical convergence order in the spatial direction is $\mathcal{O}(h^4)$, as in Theorem 3.2.

Table 1: Maximum error and convergence order in the time direction at $t = 1$ for Example 4.1, with $h = 1/10^4$.

N	$\alpha = 1.9$		$\alpha = 1.8$		$\alpha = 1.7$		$\alpha = 1.6$	
	$E_\infty(h,\tau)$	order	$E_\infty(h,\tau)$	order	$E_\infty(h,\tau)$	order	$E_\infty(h,\tau)$	order
40	4.6883e-3	1.075	3.0628e-3	1.176	1.5965e-3	1.287	6.9559e-4	1.404
80	2.2258e-3	1.088	1.3558e-3	1.189	6.5427e-4	1.298	2.6279e-4	1.417
160	1.0473e-3	1.094	5.9452e-4	1.197	2.6614e-4	1.307	9.8441e-5	1.448
320	4.9056e-4	1.098	2.5937e-4	1.202	1.0756e-4	1.323	3.6140e-5	1.528
640	2.2921e-4		1.1272e-4		4.2985e-5		1.2532e-5	

Table 2: Maximum error and convergence order in the spatial direction at $t = 1$ for Example 4.1, with $\tau = 1/(4\cdot 10^5)$.

M	$\alpha = 1.9$		$\alpha = 1.8$		$\alpha = 1.7$		$\alpha = 1.6$	
	$E_\infty(h,\tau)$	order	$E_\infty(h,\tau)$	order	$E_\infty(h,\tau)$	order	$E_\infty(h,\tau)$	order
2	7.1625e-3	4.107	5.7518e-3	4.116	4.6717e-3	4.125	3.9375e-3	4.130
4	4.1570e-4	4.011	3.3182e-4	4.060	2.6784e-4	4.132	2.2489e-4	4.205
8	2.5787e-5	3.973	1.9896e-5	4.596	1.5275e-5	6.258	1.2195e-5	3.869
16	1.6423e-6		8.2246e-7		1.9965e-7		8.3460e-7	

Example 4.2. (Non-homogeneous PDE)

$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} + \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \qquad 0 < x < 1, \quad 0 < t \le 1\,,\\
&u(x,0) = 0, \qquad \frac{\partial u(x,0)}{\partial t} = 0, \qquad 0 \le x \le 1\,,\\
&u(0,t) = 0, \qquad u(1,t) = 0, \qquad 0 < t \le 1\,,
\end{aligned}$$

where $f(x,t) = \left[(\alpha+1)t^\alpha + \Gamma(\alpha+2)t + \pi^2 t^{\alpha+1}\right]\sin(\pi x)$, with the solution

$$u(x,t) = t^{\alpha+1}\sin(\pi x).$$

The maximum error and convergence order in the time and spatial directions are reported in Tables 3 and 4, respectively. The numerical results are again consistent with Theorem 3.2.

Table 3: Maximum error and convergence order in the time direction at $t = 1$ for Example 4.2, with $h = 1/10^3$.

N	$\alpha = 1.9$		$\alpha = 1.8$		$\alpha = 1.7$		$\alpha = 1.6$	
	$E_\infty(h,\tau)$	order	$E_\infty(h,\tau)$	order	$E_\infty(h,\tau)$	order	$E_\infty(h,\tau)$	order
800	2.2021e-4	1.097	7.5406e-5	1.193	2.4411e-5	1.281	7.3324e-6	1.349
1600	1.0297e-4	1.098	3.2991e-5	1.196	1.0045e-5	1.289	2.8779e-6	1.367
3200	4.8093e-5	1.099	1.4402e-5	1.198	4.1122e-6	1.293	1.1154e-6	1.379
6400	2.2450e-5		6.2796e-6		1.6783e-6		4.2889e-7	

Table 4: Maximum error and convergence order in the spatial direction at $t = 1$ for Example 4.2, with $\tau = 1/(5 \cdot 10^4)$.

M	$\alpha = 1.9$		$\alpha = 1.8$		$\alpha = 1.7$		$\alpha = 1.6$	
	$E_\infty(h,\tau)$	order	$E_\infty(h,\tau)$	order	$E_\infty(h,\tau)$	order	$E_\infty(h,\tau)$	order
2	1.0659e-2	4.078	1.1905e-2	4.084	1.3083e-2	4.087	1.4170e-2	4.089
4	6.3117e-4	3.946	7.0207e-4	4.009	7.7008e-4	4.022	8.3279e-4	4.025
8	4.0964e-5		4.3618e-5		4.7399e-5		5.1160e-5	

5. Concluding Remarks

A numerical solution of the generalized Cattaneo equation (1.1) has been considered, by applying the L_1 approximation to the time differential part and a compact finite difference scheme to the spatial derivative. Theoretical analysis shows that the proposed scheme has $\mathcal{O}(h^4)$ accuracy in the spatial direction and $\mathcal{O}(\tau^{3-\alpha})$ accuracy in the time. Our difference scheme is more accurate in the spatial direction than either of the two schemes proposed in Ref. [5], which only provide second order accuracy. In the time direction, our scheme is also more accurate than the fully implicit scheme in Ref. [5], which only provides first order accuracy — and it is competitive with their explicit predictor-corrector scheme, which has α-th order accuracy. Finally, we remark that after completing this paper we found Ref. [3], where the tools used are much the same and the results are similar ($\mathcal{O}(h^4 + \tau^{3-\alpha})$). However, the fractional diffusion-wave equation considered in Ref. [3] seems to be a special case of the generalized Cattaneo equation considered here. Future work could concentrate on studying the superconvergence property of numerical schemes for (1.1), following an idea proposed in Ref. [22]; and on improving the convergence rate in the time direction — e.g. by employing a “graded mesh" [24] to recover the optimal convergence rate (essentially by removing $-\alpha$ in Theorem 3.5).

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