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L-Factors and Adjacent Vertex-Distinguishing Edge-Weighting

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Abstract. An edge-weighting problem of a graph G is an assignment of an integer weight to each edge e. Based on an edge-weighting problem, several types of vertex-coloring problems are put forward. A simple observation illuminates that the edge-weighting problem has a close relationship with special factors of the graphs. In this paper, we generalise several earlier results on the existence of factors with pre-specified degrees and hence investigate the edge-weighting problem — and in particular, we prove that every 4-colorable graph admits a vertex-coloring 4-edge-weighting.

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1. Introduction

In this paper, we consider only finite, undirected and simple graphs. For a graph $G = (V, E)$, if $v \in V(G)$ and $e \in E(G)$ let $v \sim e$ mean that v is an end-vertex of e. For $v \in V(G)$, $N_G(v)$ denotes the set of vertices adjacent to v. For a spanning subgraph H of G and $W \subseteq V(G)$, we use $d_H(v)$ for the number of neighbors of v in H and $d_W(v) = |N_G(v) \cap W|$. In addition, let $\omega(H)$ denote the number of connected components of H. A *k-vertex coloring* c of G is an assignment of k integers, $1, 2, \cdots, k$, to the vertices of G, and the color of a vertex v is denoted by $c(v)$. The coloring is *proper* if no two adjacent vertices share the same color. A graph G is *k-colorable* if G has a proper k-vertex coloring. The *chromatic number* $\chi(G)$ is the minimum number r such that G is r-colorable. For integers a and b, $[a, b]$ denotes the integers n with $a \leq n \leq b$. Notations and terminologies that are not defined here may be found in Ref. [6].

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A *k-edge-weighting* of a graph G is an assignment $w: E(G) \to \{1, \cdots, k\}$. An edge-weighting naturally induces a vertex coloring $c(u)$ by defining $c(u) = \sum_{u\sim e} w(e)$ for every vertex $u \in V(G)$. A k-edge-weighting of a graph G is *vertex-coloring* if the induced vertex-coloring is proper, i.e. $c(u) \neq c(v)$, and we say G admits a *vertex-coloring k-edge-weighting*.

A k-edge-weighting can also be viewed as a partition of edges into sets $\{S_1, \cdots, S_k\}$. For each vertex v, let X_v denote the multiset where the elements are the weightings of the edges incident with v, and multiplicity of i in X_v which is the number of edges incident to v in S_i. An edge-weighting is *proper* if no two incident edges receive the same label. An edge-weighting is *adjacent vertex-distinguishing* if for every edge $e = uv$, $X_u \neq X_v$; it is *vertex-distinguishing* if $X_u \neq X_v$ holds for any pair of vertices $u, v \in V(G)$. Proper (adjacent) vertex-distinguishing edge-weighting has been studied by many researchers [4,5,7], and is reminiscent of harmonious colorings [10]. Clearly, if a k-edge-weighting is vertex-coloring, then it is adjacent vertex-distinguishing. However, the converse may not hold.

If a graph has an edge as a component, it cannot have an adjacent vertex-distinguishing or vertex-coloring edge -weighting. Thus in this paper we only consider graphs without an edge component, and refer to them as *nice graphs*. The initial study of vertex-coloring and adjacent vertex-distinguishing edge-weightings posed the following conjecture.

Conjecture 1.1. (Karoński *et al.* [13]) Every nice graph admits a vertex-coloring 3-edge-weighting.

Furthermore, Karoński *et al.* proved that this conjecture holds for 3-colorable graphs (Theorem 1 of [13]). Chang *et al.* [8] considered bipartite graphs $G = (X, Y)$, and proved that if $|X||Y|$ is even the graph admits a vertex-coloring 2-edge-weighting. Lu *et al.* [9] improved this result, by showing that all 3-connected bipartite graphs have vertex-coloring 2-edge-weighting. For general graphs, Addario-Berry *et al.* [2] showed that every nice graph admits a vertex-coloring 30-edge-weighting. Addario-Berry *et al.* [3] then proved that every nice graph permits a vertex-coloring 16-edge-weighting. Wang and Yu [15] improved this bound to 13, and Kalkowski *et al.* [12] proved that every nice graph permits a vertex-coloring 5-edge-weighting.

On the other hand, there are many results for adjacent vertex-distinguishing edge-weighting. Every nice graph permits an adjacent vertex-distinguishing 213-edge-weighting and graphs with minimum degree at least 10^{99} permit an adjacent vertex-distinguishing 30-edge-weighting [13]; and every nice graph permits an adjacent vertex-distinguishing 4-edge-weighting, and that graphs of minimum degree at least 1000 permit an adjacent vertex-distinguishing 3-edge-weighting [1].

There is a close relationship between 2-edge-weighting and a special list factor. If $L: V(G) \mapsto 2^N$ is a set function, a *list factor* (or *L-factor* for short) of a graph G is a spanning subgraph H such that $d_H(v) \in L(v)$ for all $v \in V(G)$.

In general, an L-factor problem is NP-complete, even when G is bipartite. A comprehensive investigation of L-factors was carried out by Lovász [14]. An L-factor is a spanning subgraph with degrees from specified sets. When each $L(v)$ is an interval, L-factors are the same as usual degree factors. For instance, let f and g be nonnegative integer-valued functions on $V(G)$ with $f \geq g$ and $L(v) = [g(v), f(v)]$ for $v \in V(G)$. An L-factor is then exactly a (g,f)-factor. It has been shown that every graph has a spanning subgraph where every vertex has a pre-specified degree [2,3]. In Section 2, we generalize earlier results [2,3] about L-factors. Then using these results, in Section 3 we show that nice graphs with appropriate degree condition have an adjacent vertex-distinguishing 2-edge-weighting.

Weighting the edges of a graph with elements of a group Γ also gives rise to a vertex-coloring. If the vertex-coloring is proper, we say that G admits a *vertex-coloring* Γ*-edge-weighting*. The edge-weighting problem on groups has been studied by Karoński *et al.* [13], who proved that if Γ is a finite abelian group of odd order and G is a non-trivial $|\Gamma|$-colorable graph then G admits a vertex-coloring Γ-edge-weighting. In Section 4, we obtain several results on vertex-coloring Γ-edge-weighting, and consequently deduce that every 4-colorable graph admits a vertex-coloring 4-edge-weighting.

2. Subgraphs with Pre-Specified Degree

There are the following previous results about L-factors.

Theorem 2.1. (Addario-Berry *et al.* [2]) *Let G be a graph and $L(v)=\{a_v^-,a_v^-+1,a_v^+,a_v^++1\}$, for every $v\in V(G)$ such that $d_G(v)/3\le a_v^-\le d_G(v)/2\le a_v^+\le 2d_G(v)/3$. Then G contains an L-factor.*

Theorem 2.2. (Addario-Berry *et al.* [3]) *Let G be a graph and $L(v)=\{a_v^-,a_v^-+1,a_v^+,a_v^++1\}$, for every $v\in V(G)$ such that $a_v^-\le\lfloor d(v)/2\rfloor\le a_v^+<d(v)$, and*

$$a_v^+\le\min\left\{\frac{1}{2}(d(v)+a_v^-)+1,2(a_v^-+1)+1\right\}.$$

Then G contains an L-factor.

Theorem 2.3. (Addario-Berry *et al.* [3]) *Let $G=(X,Y)$ be a bipartite graph. For $v\in X$, let $a_v^-=\lfloor d(v)/2\rfloor$ and $a_v^+=a_v^-+1$. For $v\in Y$, choose a_v^-, a_v^+ such that $a_v^-\le d(v)/2\le a_v^+$ and $a_v^+\le\min\{(d(v)+a_v^-)/2+1,2a_v^-+1\}$. Let $L(v)=\{a_v^-,a_v^+\}$ for every vertex $v\in V$. Then G contains an L-factor.*

The following characterization of (g,f)-factors was given by Heinrich *et al.* [11] for the case where $g<f$.

Theorem 2.4. (Heinrich *et al.*, [11]) *Let $G=(V,E)$ be a graph and, for all $v\in V$, integers a_v, b_v such that $0\le a_v\le b_v\le d(v)$. Assume that one of the following two conditions holds:*

(a) $a_v<b_v$ for all $x\in V$;

(b) G is bipartite.

Then G has a (a_v,b_v)-factor if and only if for all disjoint sets of vertices of A and B,

$$\sum_{v\in A}(a_v-d_{G-B}(v))\le\sum_{v\in B}b_v\,. \tag{2.1}$$

We now extend the proof techniques in Refs. [2] and [3] to obtain the following more general results.

Theorem 2.5. *Suppose that c_1, c_2 and c_3 are three constants, where $0 < c_1 < c_2 < c_3 < 1$, $c_3 - \frac{c_1(1-c_3)}{1-c_1} \le c_2$ and $c_1 \ge c_3 c_2$. Let G be a graph and $L(v) = \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}$ for every vertex $v \in V(G)$ such that $c_1 d_G(v) \le a_v^- \le c_2 d_G(v) \le a_v^+ \le c_3 d_G(v)$. Then G contains an L-factor.*

Proof. Given a set of integers $S = \{a_v \mid v \in V\}$ and a subgraph H of G, we define the *deficiency* of H with respect to S to be

$$def(H) = \sum_{v \in V(G)} \max\{0, a_v - d_H(v)\} .$$

Suppose that G contains no L-factor. Choose $a_v \in \{a_v^-, a_v^+\}$, $b_v = a_v + 1$ and a spanning subgraph H of G such that for all $v \in V, d_H(v) \le b_v$, so that the deficiency is minimized over all such choices. Necessarily, there exists at least one vertex $v \in V$ such that $d_H(v) < a_v$, so the deficiency of H is positive.

Let $A_0 = \{v \mid d_H < a_v\}$. An H-alternating trail is a trail $P = v_0 v_1 \cdots v_k$ with $v_0 \in A_0$ and $v_i v_{i+1} \notin H$ for i even, $v_i v_{i+1} \in H$ for i odd. Let

$$A = \{v \mid \text{there is an even } H\text{-alternating trail ending at } v\},$$

and

$$B = \{v \mid \text{there is an odd } H\text{-alternating trail ending at } v\}.$$

Note that $A_0 \subseteq A$. For $v \in A$, $d_H(v) \le a_v$; or else, by alternating the edges in H along an even alternating trail ending in v we obtain a subgraph with less deficiency. Similarly, for $v \in B$, $d_H(v) = b_v$; or else we can likewise decrease the deficiency by alternating the edges of H, this time along an odd alternating trail ending at v. Note that $b_v > a_v$ implies that A and B are disjoint. Furthermore, for any edge e with one end in A and the other end not in B, $e \in E(H)$; for any edge e with one end in B and the other end not in A, $e \notin E(H)$. From these observations,

$$\sum_{v \in A} a_v > \sum_{v \in A} d_H(v) = \sum_{v \in B} d_H(v) + \sum_{v \in A} d_{G-B}(v) = \sum_{v \in A} d_{G-B}(v) + \sum_{v \in B} b_v \, , \tag{2.2}$$

which implies that (a) of Theorem 2.4 fails for the sets A and B.

We make two claims:

$$\text{for all } v \in A, a_v - d_{G-B}(v) \le c_2 d_B(v) \tag{2.3}$$

and

$$\text{for all } v \in B, b_v \ge c_2 d_A(v) \, . \tag{2.4}$$

Then (2.3) and (2.4) together with the fact $\sum_{v\in A} d_B(v) = \sum_{v\in B} d_A(v)$ imply inequality (2.1) holds for the sets A and B, a contradiction to (2.2). So to complete the proof, we only need to prove (2.3) and (2.4).

To see (2.3), we consider $v \in A$. Assume that $d_H(v) < a_v$. (Note that alternating edges in H along an even alternating trail does not change the deficiency and the sets A and B. Thus we assume that any vertex $v \in A$ satisfies $d_H(v) < a_v$.) Furthermore, we may assume $a_v = a_v^+ \geq c_2 d(v)$, or else (2.3) holds automatically. We may also assume that $d_{G-B}(v) > a_v^- + 1$; otherwise, by setting $a_v = a_v^-$ and removing some edges in H from v to B, we can reduce the deficiency. Thus

$$d_{G-B}(v) = d_{H-B}(v) > a_v^- + 1 \geq c_1 d(v)\,,$$

which implies $d_B(v) < (1-c_1)d(v)$ and hence $d_{G-B}(v) > \frac{c_1}{(1-c_1)} d_B(v)$. Consequently,

$$\begin{aligned} a_v - d_{G-B}(v) &\leq c_3 d(v) - d_{G-B}(v) \\ &= c_3 d_B(v) - (1-c_3) d_{G-B}(v) \\ &< c_3 d_B(v) - (1-c_3)\frac{c_1}{(1-c_1)} d_B(v) \\ &\leq c_2 d_B(v)\,. \end{aligned}$$

Next we show (2.4). Let $v \in B$. We may assume that $b_v = a_v^- + 1 < c_2 d(v)$, or else (2.4) holds trivially. Suppose that the statement fails, so $d_A(v) > b_v/c_2 \geq (c_1 d(v)+1)/c_2 \geq (c_1 d(v))/c_2 + 1/c_2 > a_v^+$. There are $d_A(v) - b_v$ edges from v to A that are not in H. In particular, there is a $w \in N(v) \cap A$ such that $vw \notin H$. As noted above, we can ensure that $d_H(w) < a_w$. This will not change the fact that $vw \notin H$. Setting $a_v = a_v^+$ and adding $a_v^+ - d_H(v)$ edges from v to A into H (including the edge vw), we decrease the deficiency. ☐

Remark 2.1. Theorem 2.5 is a generalization of Theorem 2.1. To see this, let $c_1 = 1/3$, $c_2 = 1/2$, $c_3 = 2/3$, when c_1, c_2 and c_3 satisfy the conditions in Theorem 2.5.

The following result is another extension of Theorem 2.1, where we consider three consecutive pairs in $L(v)$.

Theorem 2.6. *For every vertex v of graph G, suppose that we have chosen three integers a_v^1, a_v^2, a_v^3 such that $\frac{3}{10} d_G(v) \leq a_v^1 \leq \frac{4}{10} d_G(v) \leq a_v^2 \leq \frac{6}{10} d_G(v) \leq a_v^3 \leq \frac{7}{10} d_G(v)$. Let $L(v) = \{a_v^1, a_v^1+1, a_v^2, a_v^2+1, a_v^3, a_v^3+1\}$ for every $v \in V(G)$. Then G contains an L-factor.*

Proof. Suppose the theorem does not hold. By Theorem 2.5, choose $a_v \in \{a_v^1, a_v^2, a_v^3\}$, $b_v = a_v + 1$ and a spanning subgraph H of G with $d_H(v) \leq b_v$ for all $v \in V$, so that the deficiency is minimized over all such choices. We construct A and B as in the proof of Theorem 2.5, and prove the following two claims:

$$\text{for all } v \in A, a_v - d_{G-B}(v) \leq d_B(v)/2 \tag{2.5}$$

and

$$\text{for all } v \in B, b_v \geq d_A(v)/2\,. \tag{2.6}$$

If (2.5) and (2.6) hold, then we have

$$\sum_{v\in A}(a_v - d_{G-B}(v)) \leq \frac{1}{2}\sum_{v\in A} d_B(v) = \frac{1}{2}\sum_{v\in B} d_A(v) \leq \sum_{v\in B} b_v\,,$$

a contradiction. So it remains to prove (2.5) and (2.6).

To see (2.5), consider $v \in A$, and assume $d_H(v) < a_v$. We may assume $a_v \in \{a_v^2, a_v^3\}$, or else (2.5) holds trivially. If $a_v = a_v^3$, then we assume that $d_{G-B}(v) > a_v^2 + 1$, or else by letting $a_v = a_v^2$ and removing from H some of the edges from v to B we can reduce the deficiency. Moreover, $d_{H-B} = d_{G-B} > a_v^2 + 1$, as otherwise letting $a_v = a_v^2$ and deleting edges of H between v and B contradicts our choice of H. Thus

$$d_{H-B}(v) = d_{G-B}(v) > a_v^2 + 1 \geq \frac{4}{10} d_G(v)\,,$$

which implies $d_B(v) < \frac{3}{5}d(v)$ and hence $\frac{2}{3}d_B(v) < d_{G-B}(v)$. So we have

$$\begin{aligned} a_v - d_{G-B}(v) &\leq \frac{7}{10}d(v) - d_{G-B}(v) \\ &= \frac{7}{10}d_B(v) - \frac{3}{10}d_{G-B}(v) \\ &< \frac{7}{10}d_B(v) - \frac{3}{10} * \frac{2}{3}d_B(v) \\ &\leq \frac{1}{2}d_B(v)\,. \end{aligned}$$

With similar arguments, (2.5) holds for the case of $a_v = a_v^2 \geq d_G(v)/2$.

Next, we show (2.6). Let $v \in B$. We may assume that $b_v = a_v^i + 1$, where $i = 1$ or 2, for otherwise (2.6) holds trivially. If $b_v = a_v^1 + 1$ and $d_A(v) > 2b_v$, there are $d_A(v) - b_v$ edges from v to A that are not in H, and in particular there is a vertex $w \in N(v) \cap A$, $vw \notin H$. As noted above, we can ensure that $d_H(w) < a_w$. This does not change the fact that $vw \notin H$. Setting $a_v = a_v^2$ and adding $a_v^2 - d_H(v)$ edges from v to A into H (including the edge vw), we decrease the deficiency. If $a_v = a_v^2$, the arguments are similar. □

With slight changes in the proof techniques of Theorems 2.2 and 2.3, we are able to obtain the following two generalizations (Theorems 2.2 and 2.3 corresponding to the case $c = 1/2$).

Theorem 2.7. *Let G be a graph and c be a constant satisfing $0 < c < 2/3$. For all $v \in V(G)$, given integers a_v^-, a_v^+ such that $a_v^- \leq cd(v) \leq a_v^+ < d(v)$, and*

$$a_v^+ \leq \min\{cd(v) + (1-c)a_v^- + 1, (a_v^- + 1)/c + 1\}\,. \tag{2.7}$$

Let $L(v) = \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}$. Then G contains an L-factor.

Proof. Similar to the proofs of Theorems 2.5 and 2.6, it is sufficient to prove the following two claims:

$$a_v - d_{G-B}(v) \le c d_B(v) \ \text{ for all } v \in A \tag{2.8}$$

and

$$b_v \ge c d_A(v) \ \text{ for all } v \in B\ . \tag{2.9}$$

These two statements together with the fact that $\sum_{v\in A} d_B(v) = \sum_{v\in B} d_A(v)$ imply (1) holds for the sets A and B, completing the proof of Theorem 2.7 by contradiction.

To see (2.8), consider $v \in A$ and assume $d_H(v) < a_v$. We may assume $a_v = a_v^+$, for otherwise (2.8) holds trivially. If $a_v = a_v^-$, we assume that $d_{G-B}(v) > a_v^- + 1$; or else, by letting $a_v = a_v^-$ and removing from H some of the edges from v to B, we can reduce the deficiency. Moreover, $d_{H-B} = d_{G-B} > a_v^- + 1$. Now we have

$$\begin{aligned} a_v &\le cd(v) + (1-c)a_v^- + 1 \\ &\le cd(v) + (1-c)(d_{G-B}(v) - 2) + 1 \\ &= cd_B(v) + d_{G-B}(v) + 2c - 1\ . \end{aligned}$$

Since a_v is an integer and $2c - 1 < 1/3$, then (2.8) holds.

To prove (2.9), consider any $v \in B$. We may assume $a_v = a_v^- < cd(v)$, for otherwise the statement holds trivially. Suppose that the statement fails, then $d_A(v) > (a_v + 1)/c$, thus $d_A(v) \ge a_v^+$ by (2.7). There are $d_A(v) - b_v$ edges from v to A that are not in H. In particular, there is a vertex $w \in N(v) \cap A, vw \notin H$. As noted above, we can ensure that $d_H(w) < a_w$, which does not change the fact that $vw \notin H$. Setting $a_v = a_v^+$ and adding $a_v^+ - d_H(v)$ edges from v to A into H (including the edge vw), then the deficiency decreases. □

Theorem 2.8. *Let c be a constant with $0 < c < 2/3$ and $G = (X, Y)$ be a bipartite graph. For $v \in X$, let $a_v^- = \lfloor cd(v) \rfloor$ and $a_v^+ = a_v^- + 1$. For $v \in Y$, choose a_v^-, a_v^+ such that $a_v^- \le \lfloor cd(v) \rfloor \le a_v^+$ and $a_v^+ \le \min\{cd(v) + (1-c)a_v^- + 1, a_v^-/c + 1\}$. Let $L(v) = \{a_v^-, a_v^+\}$ for all $v \in V(G)$. Then G contains an L-factor.*

Proof. As in the proof of Theorem 2.5, for a given set of choices of a_v^- and a_v^+ suppose that the theorem does not hold. Choose $b_v = a_v \in \{a_v^-, a_v^+\}$ for $v \in Y$ and a spanning subgraph H of G such that for all $v \in V, d_H(v) \le b_v$, so that the deficiency is minimized over all such choices. If A_0, A and B are defined as in Theorem 2.5, it readily follows that the bipartiteness condition implies that A and B are indeed disjoint. All the results in which edges are in and not in H from Theorem 2.5 clearly hold in this setting. Let $A_X = A \cap X$ and define A_Y, B_X and B_Y similarly. As above, for $v \in A_X$, $d_H(v) = a_v^-$ and for $v \in B_X$, $d_H(v) = a_v^+$. We have either

$$\sum_{v\in A_X} (a_v - d_{G-B_Y}(v)) - \sum_{v\in B_Y} a_v > 0 \tag{2.10}$$

or

$$\sum_{v\in A_Y}(a_v - d_{G-B_X}(v)) - \sum_{v\in B_X} a_v > 0\,, \tag{2.11}$$

since there are no edges from A_X to B_X or from A_Y to B_Y, and the negations of these two relations imply that the deficiency is in fact zero. We next show that neither of these relations hold, and thus prove the theorem.

The proof now parallels that of Theorem 2.7. Let $v \in A_X$. By the definition of a_v^-, we have

$$a_v^- - d_{G-B_Y}(v) \le cd(v) - d_{G-B_Y}(v) \le cd_{B_Y}(v)\,.$$

For $v \in B_Y$, we claim $a_v \ge cd_{A_X}(v)$. This is clear if $a_v = a_v^+$, so we may assume $a_v = a_v^-$. Suppose that our claim does not hold, so $a_v^-/c < d_{A_X}(v)$. We may set $a_v = a_v^+$ and add some edges from v to A_X into H to reduce the deficiency, contradicting its minimality. Thus

$$\sum_{v\in A_X}(a_v - d_{G-B_Y}(v)) \le e(A_X, B_Y) \le \sum_{v\in B_Y} a_v\,,$$

and (2.10) does not hold. A similar proof shows (2.11) does not hold, to complete the proof. □

3. Adjacent Vertex-Distinguishing 2-Edge-Weighting

It has been proven that every 3-colorable graph has a vertex-coloring 3-edge-weighting, and In particular it has an adjacent vertex-distinguishing 3-edge-weighting [13]. A natural question is whether every 2-colorable (bipartite) graph has an adjacent vertex-distinguishing 2-edge-weighting, and there are some previous results as follows.

Lemma 3.1. (Chang *et al.* [8]) *A non-trivial connected bipartite graph $G = (U, W)$ admits a vertex-coloring 2-edge-weighting if one of following conditions holds:*

(1) *$|U|$ or $|W|$ is even;*

(2) *$\delta(G) = 1$;*

(3) *$\lfloor d(u)/2 \rfloor + 1 \ne d(v)$ for any edge $uv \in E(G)$.*

Recently, Lu *et al.* [9] improved this result, by proving that every 3-connected bipartite graph admits a vertex-coloring 2-edge-weighting. A graph admitting a vertex-coloring k-edge-weighting has an adjacent vertex-distinguishing k-edge-weighting. Let us now consider adjacent vertex-distinguishing 2-edge-weighting on bipartite graphs and prove the following results.

Theorem 3.1. *Given a nice bipartite graph $G = (U, W)$, if there exists a vertex $v \in V(G)$ such that $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$ then G admits an adjacent vertex-distinguishing 2-edge-weighting.*

Proof. If $|U| \cdot |W|$ is even, by Lemma 3.1 the result follows. Thus let us assume that both $|U|$ and $|W|$ are odd. For $v \in U$ such that $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$, from Lemma 4.2, G has a vertex-coloring 2-edge-weighting such that $c(x)$ is odd for all $x \in U - v$ and $c(y)$ is even for all $y \in W \cup \{v\}$. Since $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$, then G admits an adjacent vertex-distinguishing 2-edge-weighting, so we complete the proof. □

Theorem 3.2. *Every nice bipartite graph with $\delta(G) \geq 6$ admits an adjacent vertex-distinguishing 2-edge-weighting.*

Proof. Let $G = (U, W)$ be a bipartite graph. For $v \in U$, let $a_v^- = \lfloor d(v)/2 \rfloor$ and $a_v^+ = a_v^- + 1$. For $v \in W$, choose $a_v^- = \lfloor d(v)/2 \rfloor - 1$ and $a_v^+ = \lfloor d(v)/2 \rfloor + 2$. Since $\delta(G) \geq 6$, in W we have a_v^- and a_v^+ satisfying the condition of Theorem 2.3 — i.e. $a_v^+ \leq min\{(d(v) + a_v^-)/2 + 1, 2a_v^- + 1\}$, so there is a spanning subgraph H such that $d_H(v) \in \{\lfloor d(v)/2 \rfloor, \lfloor d(v)/2 \rfloor + 1\}$ for all $v \in U$, $d_H(v) \in \{\lfloor d(v)/2 \rfloor - 1, \lfloor d(v)/2 \rfloor + 2\}$ for $v \in W$. Thus we can label the edges in $E(H)$ with 1 and edges in $G - E(H)$ with 2, to yield an adjacent vertex-distinguishing 2-edge-weighting of G. □

For non-bipartite graphs, Addario-Berry *et al.* [3] proved that a graph G with $\delta(G) \geq 12\chi(G)$ admits a vertex-coloring 2-edge-weighting. In [9], a lower bound of minimum degree is improved to $8\chi(G)$ — i.e., $\delta(G) \geq 8\chi(G)$, to ensure an adjacent vertex-distinguishing 2-edge-weighting in G.

4. Vertex-Coloring Γ-Edge-Weighting on Graph

In this Section, we consider the edge-weighting problem on groups, where there are already the following technical results.

Lemma 4.1. (Karoński *et al.* [13]) *Let Γ be a finite Abelian group of odd order and G a non-trivial $|\Gamma|$-colorable graph. Then there is a weighting of the edges of G with the elements of Γ such that the induced vertex weighting is proper coloring.*

Lemma 4.2. (Lu *et al.* [9]) *Let G be a connected nice graph with chromatic number $k \geq 3$ and $\Gamma = \{g_1, g_2, \cdots, g_k\}$ be a finite abelian group, where $k = |\Gamma|$. Let c_0 be any k-vertex coloring of G with color classes $\{U_1, \cdots, U_k\}$, where $|U_i| = n_i$ for $1 \leq i \leq k$. If there exists an element $h \in \Gamma$ such that $n_1 g_1 + \cdots + n_k g_k = 2h$, then there is an edge-weighting of G with the elements of Γ such that the induced vertex coloring is c_0.*

Using Lemma 4.2, we can prove the following result.

Theorem 4.1. *Let Z_r with $r \equiv 0 \ (mod\ 4)$ be a cyclic group and G be a r-colorable graph. Then there exists a vertex-coloring Z_r-edge-weighting of G.*

Proof. Let $\mathscr{U} : V(G) \mapsto Z_r$ be a proper color of G with partition $(U_1, \cdots, U_r)$ and $\mathscr{U}(U_i) = i$. If $\sum_{i=1}^{r} i|U_i|$ is even, then by Lemma 4.2 the result follows. Now we assume $\sum_{i=1}^{r} i|U_i|$ is odd, so $\sum_{i=1}^{r/2} (2i-1)|U_{2i-1}|$ is odd. Since $r/2$ is even, we can assume that there exists some U_{2i-1} with even order. If there exists some U_{2j} with odd order, then we recolor U_{2j} with color $2i-1$ and U_{2i-1} with $2j$ and the remaining classes remains unchanged. Then $\sum_{i=1}^{r}\mathscr{U}(i)|U_i|$ is even and by Lemma 4.2 the result follows. Now let us assume that $|U_{2k}|$ is even for $k=1,2,\cdots,r/2$. Note that there exists a set U_{2l-1} with odd order. Now we recolor U_{2k} with color $2l-1$ and U_{2l-1} with $2k$ and the remaining classes remain unchanged, so the result follows. □

From the above Theorem, the following results can easily be deduced.

Theorem 4.2. *If G be a* 4*-colorable graph, then G admits a vertex-coloring* 4*-edge-weighting.*

Corollary 4.1. *Let G be a r-colorable graph, where $r \neq 4k+2$. Then G has a vertex-coloring r-edge-weighting.*

Since every planar graph is 4-colorable, we obtain:

Corollary 4.2. *Every planar graph admits a vertex-coloring* 4*-edge-weighting.*

Theorem 4.3. *Let G be a r-colorable graph, and suppose G does not admit a vertex-coloring Z_r-edge-weighting. If $\lambda : V(G) \mapsto Z_r$ is an arbitrary proper color of G, then $|\lambda^{-1}(i)|$ is odd for $i=1,\cdots,r$.*

Proof. By Lemma 4.1 and Theorem 4.1, we can assume $r \equiv 2 \pmod 4$. Suppose the result does not hold., so there exists a set with even order, say $\lambda^{-1}(i)$. Note that $\sum_{l=1}^{r/2}(2l-1)|\lambda^{-1}(2l-1)|$ is odd, so there exists some $l \neq i$ such that $|\lambda^{-1}(2l-1)|$ is odd. If i is even, then we recolor $\lambda^{-1}(2l-1)$ with i and color i with $2l-1$ to obtain a coloring λ'. Then $\sum_{l=1}^{r} l|\lambda'^{-1}(l)|$ is even, a contradiction to Lemma 4.2 and hence i is odd. Moreover, we can assume $|\lambda^{-1}(2l)|$ is odd for $l=1,\cdots,r/2$. Now we recolor $\lambda^{-1}(i)$ with 2 and $\lambda^{-1}(2)$ with i, and obtain a coloring λ''. Clearly $\sum_{l=1}^{r} l|\lambda''^{-1}(l)|$ is even, a contradiction once again. □

Theorem 4.4. *Let G be a k-colorable graph, where $(U_0, U_1, \cdots, U_{k-1})$ denote coloring classes of G. Then G admits a vertex-coloring k-edge-weighting, if any of following conditions holds:*

(i) $k \equiv 0 \pmod 4$;

(ii) $\delta(G) \leq k-2$;

(iii) *there exists a class U_i with $|U_i| \equiv 0 \pmod 2$ for some $i \in \{0,1,\cdots,k-1\}$;*

(iv) $|V(G)|$ *is odd.*

Proof. (i) From Lemma 4.1 and Theorem 4.1, the result follows.
(ii) Let $\lambda : V(G) \mapsto Z_r$ be a proper vertex coloring with partition $(U_1,\cdots,U_r)$. From Theorem 4.3, $|U_i|$ is odd for $i=1,\cdots,r$. Let $d_G(v) \leq r-2$ and $v \in U_i$. Clearly, there exists some U_j with $i \neq j$ such that there is no edge between v and U_j. We can re-color v with j and leave the coloring of the rest vertices unchanged to obtain a new coloring λ', in contradiction to Theorem 4.3.
(iii) From Theorem 4.3, the result is obvious.
(iv) Consider $r \equiv 2 \pmod 4$. If $|G| = \sum_{i=1}^{r}|U_i|$ is odd, then there exists some U_i such that $|U_i|$ is even. □

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