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A New Preconditioned Generalised AOR Method for the Linear Complementarity Problem Based on a Generalised Hadjidimos Preconditioner

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Abstract. A new generalised Hadjidimos preconditioner and preconditioned generalised AOR method for the solution of the linear complementarity problem are presented. The convergence and convergence rate of the new method are analysed, and numerical experiments demonstrate that it is efficient.

AMS subject classifications: 65H10, 65Y05, 65F10

Key words: Linear complementarity problem, generalised Hadjidimos preconditioner, PGAOR, M-matrix.

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1. Introduction

Many researchers have studied various preconditioners to solve the well known linear algebraic system

$$Ax = b\ ,$$

so that corresponding classical iterative methods such as Jacobi or Gauss-Seidel converge faster. Hadjidimos [10] considered the preconditioner

$$P_1(\alpha) \equiv I + S_1(\alpha) = \begin{pmatrix} 1 & & & & & \\ -\alpha_2 a_{21} & 1 & & & & \\ \vdots & & \ddots & & & \\ -\alpha_i a_{i1} & & & 1 & & \\ \vdots & & & & \ddots & \\ -\alpha_n a_{n1} & & & & & 1 \end{pmatrix}, \tag{1.1}$$

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where $\alpha = [0, \alpha_2, \cdots, \alpha_i, \cdots, \alpha_n] \in R^n$ involves constants $\alpha_i \geq 0$, $i = 2(1)n$ and

$$S_1(\alpha) = \begin{pmatrix} 0 & & & & & \\ -\alpha_2 a_{21} & 0 & & & & \\ \vdots & & \ddots & & & \\ -\alpha_i a_{i1} & & & 0 & & \\ \vdots & & & & \ddots & \\ -\alpha_n a_{n1} & & & & & 0 \end{pmatrix}. \tag{1.2}$$

In the case where $\alpha_i = 1$, $i = 2(1)n$, $P_1(\alpha)$ is the Milaszewicz preconditioner [17], which eliminates the elements of the first column of A below the diagonal.

It has been found that preconditioner modifications can improve the convergence rates of classical iterative methods [10]. Wang [11] presented a preconditioner $P = I + S_{\alpha\beta}$, where α, β are constants and

$$S_{\alpha\beta} = \begin{pmatrix} 0 & & & & & \\ 0 & 0 & & & & \\ \vdots & & \ddots & & & \\ 0 & & & 0 & & \\ \vdots & & & & \ddots & \\ -a_{n1}\alpha - \beta & 0 & & \cdots & & 0 \end{pmatrix}. \tag{1.3}$$

If $\beta = 0$, the Wang preconditioner becomes the Evans preconditioner [7]. In this paper, we extend the Hadjidimos and Wang preconditioner approach by constructing a **generalised Hadjidimos preconditioner** $P_1(\gamma\beta) = I + S_1(\gamma\beta)$, where

$$S_1(\gamma\beta) = \begin{pmatrix} 0 & & & & & \\ -\gamma_2 a_{21} - \beta_2 & 0 & & & & \\ \vdots & & \ddots & & & \\ -\gamma_i a_{i1} - \beta_i & & & 0 & & \\ \vdots & & & & \ddots & \\ -\gamma_n a_{n1} - \beta_n & & & & & 0 \end{pmatrix}, \tag{1.4}$$

$\gamma = [0, \gamma_2, \cdots, \gamma_i, \cdots, \gamma_n] \in R^n$, $\gamma_i \geq 0, i = 2(1)n$, and β_i, $i = 2(1)n$ are constants. Thus in (1.4), if $\gamma_i = 1, i = 2(1)n$, $\beta_i = 0, i = 2(1)n$, $P_1(\gamma\beta)$ we have the Milaszewicz preconditioner, and if $\gamma_i = 0, i = 2(1)n-1$, $\beta_i = 0, i = 2(1)n-1$, $P_1(\gamma\beta)$ the Wang preconditioner.

Given the established efficiency of preconditioners for solving linear algebraic systems, in this paper we consider the solution of the linear complementarity problem [13]:

find $x \in R^n$ such that

$$x \geq 0,\ Ax - f \geq 0,\ x^\top (Ax - f) = 0\,, \tag{1.5}$$

where $A = [a_{ij}] \in R^{n\times n}$ is a given matrix and $f \in R^n$ is a vector. Many solution methods have been considered [2,3,5,6,13,15,16]. We discuss a new PGAOR (preconditioned generalised AOR) to accelerate these methods for the linear complementarity problem (1.5), using the above generalised Hadjidimos preconditioner.

In Section 2, some preliminaries and the new PGAOR are presented. Convergence analysis is given in Section 3. The convergence rates of the PGAOR are compared with other preconditioner approaches in Section 4. Numerical experiments are discussed in Section 5, followed by our conclusions in Section 6.

2. Preliminaries and the New PGAOR

Let us first briefly summarise the notation. In reference to R^n and $R^{n\times n}$, the relation $\geq$ denotes partial ordering. In addition, for $x, y \in R^n$ we write $x > y$ if $x_i > y_i$, $i = 1, 2, \cdots, n$. A nonsingular matrix $A = (a_{ij}) \in R^{n\times n}$ is termed an $M-$matrix if $a_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$. Its comparison matrix $<A>= (\alpha_{ij})$ is defined by $\alpha_{ii} = |a_{ii}|$, $\alpha_{ij} = -|a_{ij}|$ $(i \neq j)$. A is said to be an $H-$matrix if $<A>$ is an $M-$matrix. For simplicity, we may assume that $a_{ii} = 1$, $i = 1(1)n$.

Let $x \in R^n$, $\left(x_+\right)_j = \max\left\{0, x_j\right\}$, $j = 1, 2, \cdots, n$ for any $x, y \in R^n$. We have that:

1) $(x + y)_+ \leq x_+ + y_+$;
2) $x_+ - y_+ \leq (x - y)_+$;
3) $|x| = x_+ + (-x)_+$; and
4) $x \leq y$ implies that $x_+ \leq y_+$.

The linear complementarity problem (1.5), conveniently denoted by $LCP(A, f)$(1.5), is equivalent to [1]

$$z = (z - \alpha E(Az + f))_+ ,$$

where α is a positive constant and the matrix E is positive diagonal. We begin with a Lemma together with its appropriate reference, a practice we also continue elsewhere if no proof is provided.

Lemma 2.1. [2] *If $A \in R^{n\times n}$ is a positive diagonal $M-$matrix, then $LCP(A, f)$ (1.5) has a unique solution $x^* \in R^n$.*

Consider $A = I - L - U$ where L and U are strictly lower and strictly upper triangular matrices, and denote $D = diag(A)$. A generalised AOR (GAOR) algorithm is [12]:

$$z^{k+1} = \left(z^k - D^{-1}\left[\alpha\Omega L z^{k+1} + (\Omega A - \alpha\Omega L) z^k + \Omega f\right]\right)_+ .$$

If $\alpha = 1$, then this GAOR is simply

$$z^{k+1} = \left(z^k - D^{-1}\left[\Omega L z^{k+1} + \Omega(A - L) z^k + \Omega f\right]\right)_+ ;$$

and if $\alpha = \gamma/\omega, \Omega = \omega I$, then

$$z^{k+1} = \left(z^k - D^{-1}(\gamma L z^{k+1} + \left(\omega A - \gamma L\right) z^k + \omega f\right)_+ .$$

With $J = D^{-1}(L+U)$, $\Omega = diag\left(\omega_1, \cdots, \omega_n\right)$, $\omega_i \in R_+$ and α a real constant, we have the following GAOR:

1) Given $z^0 \in R^n, k = 0$;
2) $z^{k+1} = \left(z^k - D^{-1}\left[\alpha\Omega L z^{k+1} + (\Omega A - \alpha\Omega L) z^k + \Omega f\right]\right)_+$;
3) If $z^{k+1} = z^k$, stop; otherwise, set $k = k+1$ and return to Step 2.

On denoting $G = I - \alpha\Omega D^{-1}|L|$ and $F = |I - D^{-1}(\Omega A - \alpha L)|$, we have

Lemma 2.2. [12] *Suppose that A is a positive diagonal $H-$matrix. Then for any initial vector $z^0 \in R^n$, the iterative sequence $\{z^k\}$ generated by the GAOR converges to the unique solution z^* of $LCP(A,f)$ (1.5), and*

$$\rho\left(G^{-1}F\right) \le \max_{1\le i\le n}\left\{\left|1-\omega_i\right| + \omega_i\rho\left(|J|\right)\right\} < 1 ,$$

where $0 < \omega_i < 2/[1+\rho(|J|)]$ *and* $0 \le \alpha \le 1$.

For our new PGAOR using the technique from Ref. [13] and the **generalised Hadjidimos preconditioner** $P_1(\gamma\beta) = I + S_1(\gamma\beta)$, we adopt

$$S_1(\gamma\beta) = \begin{pmatrix} 0 & & & & & \\ -\gamma_2 a_{21} - \beta_2 & 0 & & & & \\ \vdots & & \ddots & & & \\ -\gamma_i a_{i1} - \beta_i & & & 0 & & \\ \vdots & & & & \ddots & \\ -\gamma_n a_{n1} - \beta_n & & & & & 0 \end{pmatrix},$$

$\tilde{A} = \left(I + S_1(\gamma\beta)\right)A$, $\tilde{A} = \tilde{D} - \tilde{L} - \tilde{U}$, and $\tilde{f} = \left(I + S_1(\gamma\beta)\right)f$.

Algorithm 2.1. Preconditioned GAOR (PGAOR)

1) Given $z^0 \in R^n$, $k = 0$;
2) For $k = 1, 2, \cdots$,

$$z^{k+1} = \left(z^k - \tilde{D}^{-1}\left[\alpha\Omega\tilde{L}z^{k+1} + \left(\Omega\tilde{A} - \alpha\Omega\tilde{L}\right)z^k + \Omega\tilde{f}\right]\right)_+ ; \qquad (2.1)$$

3) If $z^{k+1} = z^k$, then stop; otherwise, set $k = k+1$ and return to Step 2.

3. Convergence Analysis

Let us now consider convergence analysis for the new algorithm.

Lemma 3.1. [13] *Let A be an $M-$matrix, and x be a solution of $LCP(A,f)$ (1.5). If $f_i > 0$, then $x_i > 0$ and therefore $\sum_{j=1}^{n} a_{ij}x_j - f_i = 0$. Moreover, if $f \leq 0$, then $x = 0$ is the solution of $LCP(A,f)$ (1.5).*

If the problem $LCP(A,f)$ (1.5) has a non-zero solution, there is at least one index k such that $f_k > 0$. Let us assume that $f_1 > 0$. From Lemma 3.1, we obtain

Lemma 3.2. *Let $\tilde{A} = P_1(\gamma\beta)A \equiv [\tilde{a}_{ij}]$, $\tilde{f} = P_1(\gamma\beta)f \equiv \tilde{f}$. If $f_1 > 0$, then $LCP(A,f)$ (1.5) is equivalent to the linear complementarity problem*

$$x \geq 0, \tilde{A}x - \tilde{f} \geq 0, x^{\top}(\tilde{A}x - \tilde{f}) = 0 \,. \tag{3.1}$$

Proof. Suppose that x is the solution to $LCP(A,f)$ (1.5). Because $f_1 > 0$, from Lemma 3.1 we have that $x_1 > 0$ and $\sum_{j=1}^{n} a_{1j}x_j - f_1 = 0$.

Thus if $i = 1$,

$$\sum_{j=1}^{n} \tilde{a}_{ij}x_j - \tilde{f}_i = \sum_{j=1}^{n} a_{ij}x_j - f_i \,;$$

and if $i \neq 1$, then

$$\begin{aligned}\sum_{j=1}^{n} \tilde{a}_{ij}x_j - \tilde{f}_i &= \sum_{j=1}^{n} \left(a_{ij} - (\gamma_i a_{i1} + \beta_i)a_{1j}\right) x_j - \left(f_i - (\gamma_i a_{i1} + \beta_i)f_1\right) \\ &= \sum_{j=1}^{n}(a_{ij}x_j - f_i) - (\gamma_i a_{i1} + \beta_i)\sum_{j=1}^{n}(a_{1j}x_j - f_1) \\ &= \sum_{j=1}^{n}(a_{ij}x_j - f_i) \,. \end{aligned} \tag{3.2}$$

Consequently, x is the solution to problem (3.1).

Conversely, let us suppose that x is the solution to problem (3.1), so that from (3.2) we have $x_1 > 0$, $\sum_{j=1}^{n} a_{1j}x_j - f_1 = 0$. Moreover, for $i \neq 1$,

$$\sum_{j=1}^{n} a_{ij}x_j - f_i = \sum_{j=1}^{n} \left(\tilde{a}_{ij} + (\gamma_i a_{i1} + \beta_i)a_{1j}\right) x_j - \left(\tilde{f}_i + (\gamma_i a_{i1} + \beta_i)f_1\right) \tag{3.3}$$

$$= \sum_{j=1}^{n}(\tilde{a}_{ij}x_j - \tilde{f}_i) + (\gamma_i a_{i1} + \beta_i)\sum_{j=1}^{n}(a_{1j}x_j - f_1) \tag{3.4}$$

$$= \sum_{j=1}^{n}(\tilde{a}_{ij}x_j - \tilde{f}_i) \,, \tag{3.5}$$

so x is the solution to $LCP(A,f)$ (1.5). □

Lemma 3.3. [10] *If* $A=(a_{ij})$ *is a nonsingular* $M-$*matrix, then*

$$a_{1i}a_{i1}<1, \quad i\neq 1\ .$$

Lemma 3.4. [18] *Let* $A=[a_{ij}]\in R^{n\times n}$*, and* $a_{ij}\leq 0$ *for* $i\neq j$*.* A *is an* $M-$*matrix if and only if there exists a positive vector* y *such that* $Ay>0$*.*

Lemma 3.5. *If* A *is an* $M-$*matrix,* $0\leq\gamma_i\leq 1$*,* $-\gamma_i a_{i1}+a_{i1}\leq\beta_i\leq-\gamma_i a_{i1}$*,* $i=2,\cdots,n$*, then* $\tilde{A}=P_1(\gamma\beta)A\equiv[\tilde{a}_{ij}]$ *is an* $M-$*matrix.*

Proof. Let $N:=\{1,2,\cdots,n\}$, $N_1:=N\setminus\{1\}$, $N_1':=\{i\in N_1|a_{i1}\neq 0\}$. Then

$$\tilde{a}_{ij}=\begin{cases} a_{ij}, & i=1,j\in N\ ;\\ (1-\gamma_i)a_{i1}-\beta_i, & i\neq 1,j=1\ ;\\ a_{ij}-(\gamma_i a_{i1}+\beta_i)a_{1j}, & i\neq 1,j\in N_1|\ . \end{cases} \tag{3.6}$$

If A is an $M-$matrix, $a_{ij}\leq 0$, $i\neq j$. From Lemma 3.3, for $0<a_{1i}a_{i1}<1$ we have that $a_{i1}>1/a_{1i}$ for $i=2,\cdots,n$. Otherwise, from $-\gamma_i a_{i1}+a_{i1}\leq\beta_i\leq-\gamma_i a_{i1}$ we have that $\beta_i+\gamma_i a_{i1}<0$, $\beta_i\geq-\gamma_i a_{i1}+a_{i1}>1/a_{1i}-\gamma_i a_{i1}$. Now if $i\neq 1$, $0\leq\gamma_i\leq 1$, then:

1. for $j=1$, $\tilde{a}_{ij}=(1-\gamma_i)a_{i1}-\beta_i\leq 0$;
2. for $j\in N_1$, $j\neq i$, $\tilde{a}_{ij}=a_{ij}-(\gamma_i a_{i1}+\beta_i)a_{1j}\leq 0$; and
3. for $j=i$, $\tilde{a}_{ii}=1-(\gamma_i a_{i1}+\beta_i)a_{1i}>1-1/a_{1i}*a_{1i}=0$.

Consequently, $\tilde{A}(\alpha)$ is an $L-$matrix. From (1.4), $P_1(\gamma\beta)=I+S_1(\gamma\beta)>0$; and from Lemma 3.4 there exists a positive vector $y>0$ such that $Ay>0$. Thus $\tilde{A}y=P_1(\gamma\beta)Ay>0$, and from Lemma 3.4 $\tilde{A}$ is an $M-$matrix. □

Theorem 3.1. *Let* A *be a diagonally dominant* $M-$*matrix. If* $0\leq\gamma_i\leq 1$*,* $-\gamma_i a_{i1}+a_{i1}\leq\beta_i\leq-\gamma_i a_{i1}$*,* $i=2,\cdots,n$*, then the iterative sequence of the algorithm 2.1 converges to the unique solution* x^* *of* $LCP(A,f)$ *(1.5).*

Proof. From Lemma 3.5, $\tilde{A}$ is a diagonally dominant $H-$matrix. Thus from Lemmas 3.2 and 2.2, Algorithm 2.1 converges to the unique solution of $LCP(A,f)$ (1.5). □

4. Comparison Theorem

We now discuss a comparison theorem that shows how our PGAOR is more efficient. Denote $\tilde{A}=P_1(\gamma\beta)A\equiv[\tilde{a}_{ij}]$, with

$$\tilde{a}_{ij}=\begin{cases} a_{ij}, & i=1,j\in N\ ;\\ (1-\gamma_i)a_{i1}-\beta_i, & i\neq 1,j=1\ ;\\ a_{ij}-(\gamma_i a_{i1}+\beta_i)a_{1j}, & i\neq 1,j\in N_1\ . \end{cases} \tag{4.1}$$

Let $\gamma_i \in [0,1]$, $i = 1,2,\cdots,n$, $A = I - L - U$,

$$D_{rw} = diag(0, (\gamma_2 a_{21} + \beta_2)a_{12}, \cdots, (\gamma_n a_{n1} + \beta_n)a_{1n}),$$

and

$$S_1(\gamma\beta)U = D_{rw} + L_{rw} + U_{rw},$$

where L_{rw} and U_{rw} are strictly lower and upper triangular matrices. From (1.4), we have that

$$S_1(\gamma\beta)U = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & (\gamma_2 a_{21} + \beta_2)a_{12} & (\gamma_2 a_{21} + \beta_2)a_{13} & \cdots & (\gamma_2 a_{21} + \beta_2)a_{1n} \\ 0 & (\gamma_3 a_{31} + \beta_3)a_{12} & (\gamma_3 a_{31} + \beta_3)a_{13} & \cdots & (\gamma_3 a_{31} + \beta_3)a_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & (\gamma_n a_{n1} + \beta_n)a_{12} & (\gamma_n a_{n1} + \beta_n)a_{13} & \cdots & (\gamma_n a_{n1} + \beta_n)a_{1n} \end{pmatrix}, \quad (4.2)$$

$$L_{rw} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & (\gamma_3 a_{31} + \beta_3)a_{12} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & (\gamma_n a_{n1} + \beta_n)a_{12} & \cdots & (\gamma_n a_{n1} + \beta_n)a_{1,n-1} & 0 \end{pmatrix},$$

and

$$U_{rw} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & (\gamma_2 a_{21} + \beta_2)a_{13} & \cdots & (\gamma_2 a_{21} + \beta_2)a_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_{n-1} a_{n-1,1} + \beta_2)a_{1n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let

$$\tilde{A} = \tilde{D} - \tilde{L} - \tilde{U},$$

where

$$\tilde{D} = I - D_{rw}, \qquad \tilde{L} = L - S_1(\gamma\beta) + L_{rw}, \qquad \tilde{U} = U + U_{rw}, \tag{4.3}$$

and

$$N := \{1,2,\cdots,n\}, \qquad N_1 := N \setminus \{1\}, \qquad N_2 := \{i \in N_1 : a_{i1} \neq 0\}. \tag{4.4}$$

Consider the following splittings [10]:

$$\tilde{A} = \begin{cases} M_1(\gamma\beta) - N_1(\gamma\beta) = (I + S_1(\gamma\beta)) - (I + S_1(\gamma\beta))(L + U), \\ M_2(\gamma\beta) - N_2(\gamma\beta) = I - (L + L_{rw} - S_1(\gamma\beta) + U + U_{rw} + D_{rw}), \\ M_3(\gamma\beta) - N_3(\gamma\beta) = (I - D_{rw}) - (L + L_{rw} - S_1(rw) + U + U_{rw}), \\ M_4(\gamma\beta) - N_4(\gamma\beta) = (I - (L - S_1(\gamma\beta))) - (D_{rw} + L_{rw} + U + U_{rw}), \\ M_5(\gamma\beta) - N_5(\gamma\beta) = (I - (L - S_1(\gamma\beta)) - L_{rw}) - (D_{rw} + U + U_{rw}), \\ M_6(\gamma\beta) - N_6(\gamma\beta) = (I - (L - S_1(\gamma\beta)) - D_{rw} - L_{rw}) - (U + U_{rw}), \end{cases} \tag{4.5}$$

and define the matrices

- $B \equiv M_1^{-1}(\gamma\beta)N_1(\gamma\beta) = L + U$;
- $B' \equiv M_2^{-1}(\gamma\beta)N_2(\gamma\beta) = L + L_{rw} + U + U_{rw} + D_{rw} - S_1(\gamma\beta)$;
- $B'' \equiv M_3^{-1}(\gamma\beta)N_3(\gamma\beta) = (I - D_{rw})^{-1}(L + L_{rw} + U + U_{rw} - S_1(\gamma\beta))$;
- $H \equiv (I - L)^{-1}U$;
- $H' \equiv M_5^{-1}(\gamma\beta)N_5(\gamma\beta) = (I - (L - S_1(\gamma\beta)) - L_{rw})^{-1}(D_{rw} + U + U_{rw})$;
- $H'' \equiv M_6^{-1}(\gamma\beta)N_6(\gamma\beta) = (I - (L - S_1(\gamma\beta)) - D_{rw} - L_{rw})^{-1}(U + U_{rw})$.

Definition 4.1. ([9,19]) If $A \in R^{n\times n}$, then

1. $A = M - N$ is a regular splitting of A if $M^{-1} \geq 0$, $N \geq 0$;
2. $A = M - N$ is an $M-$splitting of A if M is an $M-$matrix, $N \geq 0$;
3. $A = M - N$ is an $H-$compatible splitting of A if $<A>=<M> - |N|$; and
4. $A = M - N$ is convergent if the spectral radius $\rho(M^{-1}N) < 1$.

Lemma 4.1. [19] *If A is an irreducible $n \times n$ matrix, then*

1) A has a real positive spectral radius $\rho(A)$;

2) $\rho(A)$ of A corresponds to a positive eigenvector $x > 0$; and

3) $\rho(A)$ is a single eigenvalue of A .

Lemma 4.2. [19] *Let $A = (a_{ij})$ be a nonsingular $M-$matrix, and the matrix C be generated by A with some non-diagonal elements $a_{ij} = 0$, $i \neq j$. Then C is a nonsingular $M-$matrix.*

Lemma 4.3. [19] *Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geq 0$.*

(1) If $N_2 \geq N_1 \geq 0$, then
$$0 \leq \rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2) < 1 . \tag{4.6}$$

(2) If $M_1^{-1} \geq M_2^{-1}$, then
$$0 \leq \rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2) < 1 . \tag{4.7}$$

Lemma 4.4. [19] *If B is a Jacobi matrix and G is a Gauss-Seidel matrix, then the following are equivalent:*

1) $\rho(B) = \rho(G) = 0$;

2) $0 < \rho(G) < \rho(B) < 1$;

3) $\rho(B) = \rho(G) = 1$; and

4) $1 < \rho(B) < \rho(G)$.

Theorem 4.1. *Let A be an irreducible nonsingular $M-$matrix, and $0 \le \gamma_i \le 1$, $-\gamma_i a_{i1} + a_{i1} \le \beta_i \le -\gamma_i a_{i1}$ for $i = 2, \cdots, n$. Then for any $y \in R^n$, $y \ge 0$ such that*

$$B'y \le By\,, \tag{4.8}$$

$$\rho(B'') \le \rho(B') < 1\,, \tag{4.9}$$

$$\rho(H'') \le \rho(H') \le \rho(H) < 1\,, \text{ and} \tag{4.10}$$

$$\rho(H'') \le \rho(B''), \quad \rho(H') \le \rho(B'), \quad \rho(H) \le \rho(B) < 1\,. \tag{4.11}$$

Proof. From the definition of B, B' and B'',

$$b_{ii} = 0, \qquad i \in N; \qquad b_{ij} = -a_{ij}, \qquad i, j \in N, j \ne i\,.$$

$$\begin{cases} b'_{ii} = (\gamma_i a_{i1} + \beta_i)a_{1i} = (\gamma_i a_{i1} - \beta_i)a_{1i}, & i \in N_2\,; \\ b'_{ii} = 0, & i \in N \setminus N_2\,; \\ b'_{ij} = -a_{ij} = b_{ij}, & i \in N \setminus N_2, j \in N_1, j \ne i\,; \\ b'_{ij} = (\gamma_i a_{i1} + \beta_i)a_{1j} - a_{ij} = (\gamma_i b_{i1} - \beta_i)b_{1j} + b_{ij}, & i \in N_2, j \in N_1, j \ne i\,; \\ b'_{i1} = (-1 + \gamma_i)a_{i1} + \beta_i = (1 - \gamma_i)b_{i1} + \beta_i, & i \in N_2\,. \end{cases} \tag{4.12}$$

$$\begin{aligned} & b''_{ii} = 0, && i \in N\,; \\ & b''_{ij} = -a_{ij} = b_{ij}, && i \in N \setminus N_2, j \in N_1, j \ne i\,; \\ & b''_{ij} = \frac{(\gamma_i a_{i1} + \beta_i)a_{1j} - a_{ij}}{1 - (\gamma_i a_{i1} + \beta_i)a_{1i}} = \frac{(\gamma_i b_{i1} - \beta_i)b_{1j} + b_{ij}}{1 - (\gamma_i b_{i1} - \beta_i)b_{1i}}, && i \in N_2, j \in N_1, j \ne i\,; \\ & b''_{i1} = \frac{(-1 + \gamma_i)a_{i1} + \beta_i}{1 - (\gamma_i a_{i1} + \beta_i)a_{1i}} = \frac{(1 - \gamma_i)b_{i1} + \beta_i}{1 - (\gamma_i b_{i1} - \beta_i)b_{1i}}, && i \in N_2\,. \end{aligned} \tag{4.13}$$

From Lemma 4.1, for any nonnegative Jacobi iterative matrix B there exists a positive vector y such that $\rho(B)y = By$. Then

$$\begin{aligned} \rho(B)y_i = & \sum_{j \ne i} b_{ij} y_j = b_{i1} y_1 + \sum_{j \ne i, j=2}^{n} b_{ij} y_j \\ = & (b'_{i1} + \gamma_i b_{i1} - \beta_i) y_1 + \sum_{j \ne i, j=2}^{n} \left(b'_{ij} - (\gamma_i b_{i1} - \beta_i) b_{1j} \right) y_j + b'_{ii} y_i - b'_{ii} y_i \\ = & \sum_{j=1}^{n} b'_{ij} y_j - (\gamma_i b_{i1} - \beta_i) \sum_{j=2}^{n} b_{1j} y_j + (\gamma_i b_{i1} - \beta_i) b_{1i}) y_i \\ = & \sum_{j=1}^{n} b'_{ij} y_j + (\gamma_i b_{i1} - \beta_i) \left(\frac{1}{\rho(B)} - 1 \right) \sum_{j=2}^{n} b_{1j} y_j \\ \ge & \sum_{j=1}^{n} b'_{ij} y_j\,, \end{aligned}$$

so (4.8) is valid — i.e. $B'y \le By$.

From Lemma 3.5, $\tilde{A}$ is a nonsingular $M-$matrix. From (4.5), $M_2(\gamma\beta)^{-1} = I \geq 0$, $N_2(\gamma\beta) \geq 0$, $M_3(\gamma\beta)^{-1} = (I - D_{rw})^{-1} \geq 0$, $N_3(\gamma\beta) \geq 0$, they are convergent regular splittings and obviously $M_3(\gamma\beta)^{-1} \geq M_2(\gamma\beta)^{-1}$, so (4.9) follows from Lemma 4.3.

From (4.5), $M_4(\gamma\beta) = (I-(L-S_1(\gamma\beta))$ and from Lemma 4.2, $M_4(\gamma\beta)$ is a nonsingular $M-$matrix, so $M_4(\gamma\beta)^{-1} \geq 0$. Also $N_4(\gamma\beta) = D_{rw}+L_{rw}+U+U_{rw} \geq 0$, so $M_4(\gamma\beta)-N_4(\gamma\beta)$ is regular and convergent.

$M_5(\gamma\beta) = M_4(\gamma\beta) - L_{rw} = M_4(\gamma\beta)(I - M_4(\gamma\beta)^{-1}L_{rw})$. If $\bar{L} = M_4(\gamma\beta)^{-1}L_{rw}$, then

$$M_5(\gamma\beta)^{-1} = (I - \bar{L})^{-1}M_4(\gamma\beta)^{-1} = \sum_{j=1}^{n-1} \bar{L}^j M_4(\gamma\beta)^{-1} \geq 0\,,$$

so from $N_5(\gamma\beta) = D_{rw} + U + U_{rw} \geq 0$ we have $M_5(\gamma\beta) - N_5(\gamma\beta)$ regular and convergent. Similarly, we can show that $M_6(\gamma\beta) - N_6(\gamma\beta)$ is regular and convergent.

From $N_4(\gamma\beta) = D_{rw} + L_{rw} + U + U_{rw} \geq N_5(\gamma\beta) = D_{rw} + U + U_{rw} \geq N_6(\gamma\beta) = U + U_{rw}$, and from Lemma 4.3 $\rho(H'') \leq \rho(H') \leq \rho(H) < 1$.

From Lemma 4.4, $0 \leq \rho(H) < \rho(B) < 1$; and $N_2(\gamma\beta) = L - S_1(\gamma\beta) + U + U_{rw} + D_{rw} \geq N_5(\gamma\beta) = U+U_{rw}+D_{rw}$, $N_3(\gamma\beta) = L+L_{rw}-S_1(\gamma\beta)+U+U_{rw}+D_{rw} \geq N_6(\gamma\beta) = U+U_{rw}$, so from Lemma 4.3 $\rho(H') \leq \rho(B')$, $\rho(H'') \leq \rho(B'')$. □

Lemma 4.5. [4]. *If A is a nonnegative matrix,*

(a) for any vector $x \geq 0$ such that $Ax \geq \beta x$, then $\rho(A) \geq \beta$;

(b) for a vector $x > 0$ such that $Ax \leq \gamma x$, then $\rho(A) \leq \gamma$.
Moreover, if A is irreducible, $x \geq 0$ and $\beta x \leq Ax \leq \gamma x$, then $\beta < \rho(A) < \gamma$.

Theorem 4.2. *If A is an irreducible nonsingular $M-$matrix, for $i = 2, \cdots, n$, $0 \leq \gamma_i \leq 1$, $-\gamma_i a_{i1} + a_{i1} \leq \beta_i \leq -\gamma_i a_{i1}$, $0 \leq r \leq w \leq 1$ ($w \neq 0$, $r \neq 1$) we have*

$$\rho(\tilde{L}_{r,w}) \leq \rho(L_{r,w}) < 1\,, \tag{4.14}$$

where

$$\begin{aligned}\tilde{L}_{r,w} &= (\tilde{D} - r\tilde{L})^{-1}[(1-w)\tilde{D} + (w-r)\tilde{L} + w\tilde{U}]\,,\\ L_{r,w} &= (I - rL)^{-1}[(1-w)I + (w-r)L + wU]\,.\end{aligned}$$

Proof. It is obvious that $(\tilde{D} - r\tilde{L}) - [(1-w)\tilde{D} + (w-r)\tilde{L} + w\tilde{U}]$ and $(I - rL) - [(1-w)I + (w-r)L + wU]$ are regular, hence $\tilde{L}_{r,w}$ and $L_{r,w}$ are nonnegative and irreducible. Consequently, there exists a positive vector x such that

$$L_{r,w}x = \lambda x, \qquad \lambda = \rho(L_{r,w})\,,$$

so that

$$[(1-w)D + (w-r)L + wU]x = \lambda(I - rL)x\,, \tag{4.15}$$

whence

$$\tilde{L}_{r,w}x - \lambda x = (\tilde{D} - r\tilde{L})^{-1}[(1-w)\tilde{D} + (w-r)\tilde{L} + w\tilde{U} - \lambda(\tilde{D} - r\tilde{L})]x\,. \tag{4.16}$$

Substituting (4.3) and (4.15) into (4.16), we have

$$\begin{aligned}
&\tilde{L}_{r,w}x - \lambda x\\
=&(\tilde{D} - r\tilde{L})^{-1}\left[(1-w)(I - D_{rw}) + (w - r + \lambda r)(L + L_{rw} - S_1(\gamma\beta)) + wU + wU_{rw}\right]x\\
=&(\tilde{D} - r\tilde{L})^{-1}\left[(\lambda-1)D_{rw} + w(S_1(\gamma\beta)U - S_1(\gamma\beta)) + r(1-\lambda)(L_{rw} - S_1(\gamma\beta))\right]x\,.
\end{aligned}$$

Thus for $\lambda < 1$ we have $\tilde{L}_{r,w}x < \lambda x$, and the result follows from Lemma 4.5. □

5. Numerical Experiments

Results from some numerical experiments are now presented, showing that the new PGAOR is more efficient.

Example 5.1. Linear complementarity problem with coefficient matrix

$$A_1 = \begin{pmatrix} 1.00000 & -0.00580 & -0.19350 & -0.25471 & -0.03885\\ -0.28424 & 1.00000 & -0.16748 & -0.21780 & -0.21577\\ -0.24764 & -0.26973 & 1.00000 & -0.18723 & -0.08949\\ -0.13880 & -0.01165 & -0.25120 & 1.00000 & -0.13236\\ -0.25809 & -0.08162 & -0.13940 & -0.04890 & 1.00000 \end{pmatrix}.$$

Given Theorem 4.2, let us choose r, w, γ_i and β_i, and denote

$$\begin{aligned}
\tilde{L}_{r,w} &= (\tilde{D} - r\tilde{L})^{-1}\left[(1-w)\tilde{D} + (w-r)\tilde{L} + w\tilde{U}\right],\\
L_{r,w} &= (I - rL)^{-1}\left[(1-w)I + (w-r)L + wU\right].
\end{aligned}$$

We can compute the spectral radius $\rho(L_{rw})$ of L_{rw}, and the spectral radius $\rho(\tilde{L}_{rw})$ of $\tilde{L}_{rw}$. The results listed in the following tables demonstrate that the PGAOR is more efficient than the GAOR, and the generalised Hadjidimos preconditioner is more efficient than the other preconditioners. In particular, for $r = w = 1$ (the GS iterative method) the PGAOR is most efficient.

Example 5.2. Linear complementarity problem with coefficient matrix

$$A_2 = \begin{pmatrix} 1 & c_1 & c_2 & c_3 & c_1 & \dots\\ c_3 & 1 & c_1 & c_2 & \ddots & c_1\\ c_2 & c_3 & 1 & c_1 & \ddots & c_3\\ c_1 & \ddots & \ddots & 1 & \ddots & c_2\\ c_3 & \ddots & \ddots & \ddots & 1 & c_1\\ \vdots & c_3 & c_1 & c_2 & c_3 & 1 \end{pmatrix},$$

Table 1: $\rho(\tilde{L}_{rw})$ and $\rho(L_{rw})$ when $r = 0.85$, $w = 0.9$.

Preconditioner	$(0,\gamma_2,\cdots,\gamma_5)^T$	$(0,\beta_2,\cdots,\beta_5)^T$	$\rho(\tilde{L}_{rw})$	$\rho(L_{rw})$
Milaszewicz	$(0,1,1,1,1)^T$	$(0,0,0,0,0)^T$	0.4957	
Hadjidimos	$(0,1,0,0.2,1)^T$	$(0,0,0,0,0)^T$	0.4835	
Evans	$(0,0,0,0,1)^T$	$(0,0,0,0,0)^T$	0.4878	
Wang	$(0,0,0,0,1)^T$	$(0,0,0,0,0.025)^T$	0.4900	0.5086
Generalised	$(0,1,0,0,1)^T$	$(0,0,0.1,0.03,0)^T$	0.4798	
Hadjidimos	$(0,1,1,1,1)^T$	$(0,0.28,0.24,0,0)^T$	0.4859	

Table 2: $\rho(\tilde{L}_{rw})$ and $\rho(L_{rw})$ when $r = 0.95$, $w = 1$.

Preconditioner	$(0,\gamma_2,\cdots,\gamma_5)^T$	$(0,\beta_2,\cdots,\beta_5)^T$	$\rho(\tilde{L}_{rw})$	$\rho(L_{rw})$
Milaszewicz	$(0,1,1,1,1)^T$	$(0,0,0,0,0)^T$	0.3988	
Hadjidimos	$(0,1,0,0.2,1)^T$	$(0,0,0,0,0)^T$	0.3813	
Evans	$(0,0,0,0,1)^T$	$(0,0,0,0,0)^T$	0.3835	
Wang	$(0,0,0,0,1)^T$	$(0,0,0,0,0.025)^T$	0.3866	0.4117
Generalised	$(0,1,0,0,1)^T$	$(0,0,0.1,0.03,0)^T$	0.3758	
Hadjidimos	$(0,1,1,1,1)^T$	$(0,0.28,0.24,0,0)^T$	0.3805	

Table 3: $\rho(\tilde{L}_{rw})$ and $\rho(L_{rw})$ when $r = 1$, $w = 1$.

Preconditioner	$(0,\gamma_2,\cdots,\gamma_5)^T$	$(0,\beta_2,\cdots,\beta_5)^T$	$\rho(\tilde{L}_{rw})$	$\rho(L_{rw})$
Milaszewicz	$(0,1,1,1,1)^T$	$(0,0,0,0,0)^T$	0.3732	
Hadjidimos	$(0,1,0,0.2,1)^T$	$(0,0,0,0,0)^T$	0.3522	
Evans	$(0,0,0,0,1)^T$	$(0,0,0,0,0)^T$	0.3526	
Wang	$(0,0,0,0,1)^T$	$(0,0,0,0,0.025)^T$	0.3562	0.3850
Generalised	$(0,1,0,0,1)^T$	$(0,0,0.1,0.03,0)^T$	0.3455	
Hadjidimos	$(0,1,1,1,1)^T$	$(0,0.28,0.24,0,0)^T$	0.3491	

and $f = (sin(\pi(1-2x_i)))_{i=1}^n$. If we let $c_1 = -2/n$, $c_2 = 0$ and $c_3 = -1/(n+2)$, it is easy to show that A_2 is an M-matrix [14]. The initial approximation of x_0 is taken as a zero vector. The stopping criterion is $|x^T * (A_2 x - f)| \leq 10^{-8}$, and the numerical results are shown in Table 4. By using the general Hadjidimos preconditioner, the new PGAOR is evidently more efficient than the GAOR. However, it is notable that if we choose $\gamma = (0,1,0,\cdots,0,1)^T$ and $\beta = (0,0,0.45,\cdots,0.45,0)^T$, on applying the general Hadjidimos preconditioner P the coefficient matrix PA_2 is an H-matrix but not an M-matrix and the new PGAOR becomes faster. We propose to reconsider this elsewhere.

6. Conclusion

For linear systerms, preconditioners can accelerate corresponding iterative methods. In this paper, a new general Hadjidimos preconditioner is presented. Using the technique in

Table 4: Example 5.2, Comparison of iterative steps and cputimes of GAOR and PGAOR. ('-/-' mean that 'iter/cputime(second))'.

Algorithm	100	400	1000	2000	3000
GAOR	49/0.2028	81/4.8048	111/39.4059	138/116.3611	159/244.3132
PGAOR	$\gamma=(0,1,0,\cdots,0,1)^T$, $\beta=(0,0,0.999/(n+2),\cdots,0)^T$, $r=0.99, w=1$				
	48/0.1872	80/4.7892	110/38.8598	138/114.2511	158/239.3055
	$\gamma=(0,1,0,\cdots,0,1)^T$, $\beta=(0,0,0.45,\cdots,0.45,0)^T$, $r=0.99, w=1$				
	26/0.1872	41/2.6052	68/24.8822	97/82.0	122/197.1852

Ref. [13], we apply this new Hadjidimos preconditioner to establish a new PGAOR method to solve the linear complementarity problem. A comparison theorem and numerical results demonstrate the efficiency of the new method for this purpose.

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