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H^1-Stability and Convergence of the FE, FV and FD Methods for an Elliptic Equation

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Abstract. We obtain the coefficient matrices of the finite element (FE), finite volume (FV) and finite difference (FD) methods based on P_1-conforming elements on a quasi-uniform mesh, in order to approximately solve a boundary value problem involving the elliptic Poisson equation. The three methods are shown to possess the same H^1-stability and convergence. Some numerical tests are made, to compare the numerical results from the three methods and to review our theoretical results.

AMS subject classifications: 35Q30, 65N30

Key words: Finite element method, finite difference method, finite volume method, Poisson equation, stability and convergence.

1. Introduction

Elliptic equations form one of the most common classes of partial differential equations (PDE). Numerical schemes to obtain approximate solutions to elliptic equations are fundamentally different from those for parabolic and hyperbolic equations. Many practical problems involve elliptic equations — e.g. the well known cases of steady heat flow and the irrotational flow of an inviscid incompressible fluid.

Finite element (FE), finite volume (FV) and finite difference (FD) methods are three standard approaches to the discretisation of PDE that are often used for their approximate solution [1, 5, 12, 13, 17, 18]. The FV method may be regarded as a generalisation of the FD method [3, 7, 8], and may be applied to arbitrary domains without much difficulty.

The relationship between the FE and FV methods applied to the two-dimensional Poisson equation has been discussed by Vanselow [17], who also considered the FV method

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with Voronoi boxes for discretising elliptic boundary value problems — and showed that the matrix of the linear system of equations for the FV method is equivalent to the matrix for the FE method if and only if the Delaunay triangulation and P_1-conforming element are used. However, he did not prove any convergence results in comparing the FE solution and FV solutions under weaker assumptions. Mattiussi [18] applied aspects of algebraic topology to the analysis of the FV and FE methods, illustrating the similarity between the discretisation strategies adopted by the two methods via a geometric interpretation of the role played by weighting functions involved with the respective finite elements. Recently, Xu and Zou [16] presented some convergent properties of both linear and quadratic simplicial FV methods for elliptic equations. They established an inf-sup condition in a simple fashion for the linear FV method on domains of any dimension, and proved that the solution via a linear FV method is "super-close" to that from a relevant FE method — cf. also [3,4,11].

If the partition of a domain possesses certain geometrical properties, the linear convergence of the FE method with respect to a special energy norm follows when the solution of the problem involving the Poisson equation belongs to H^2 space. It is notable that the same convergence results can be obtained for the FD and FE methods based on P_1-conforming elements under weaker assumptions. In Section 2, we obtain the coefficient matrices of the FE, FD and FV methods based on continuous P_1-elements on a quasi-uniform grid to solve a boundary value problem involving the Poisson equation on a one-dimensional domain, and show that the three methods possess the same H^1-stability and convergence. The coefficient matrices of the three methods are provided and their H^1-stability and convergence are then analysed in a two-dimensional domain in Section 3. Numerical experiments are presented in Section 4, to confirm the theoretical analysis and demonstrate the numerical results of the three methods.

2. FE, FV and FD Methods in One Dimension ($d = 1$)

In this article, we consider applying FE, FD and FV methods on a quasi-uniform grid for the boundary value problem involving the Poisson equation

$$-\Delta u = f \qquad (x_1, \cdots, x_d) \in \Omega \tag{2.1}$$

$$u = 0 \qquad (x_1, \cdots, x_d) \in \partial\Omega \tag{2.2}$$

on a bounded domain $\Omega \subset R^d$ with boundary $\partial\Omega$, where $\Delta = \partial_{x_1x_1} + \cdots + \partial_{x_dx_d}$ is the Laplacian corresponding to the gradient operator $\nabla = (\partial_{x_1}, \cdots, \partial_{x_d})^\top$. We always assume $f \in L^2(\Omega)$, and the regularity estimate

$$\|u\|_{2,\Omega} \le c\|f\|_{0,\Omega} \,. \tag{2.3}$$

In this section, we consider the one-dimensional case $d = 1$, with the assumed domain $\Omega = (0, 1)$ sub-divided as $0 = x_0 < x_1 < \cdots < x_{m+1} = 1$ — i.e. into sub-intervals $I_i = x_{i-1}, x_i)$ of size $h_i = x_i - x_{i-1}$ (cf. Fig. 1). Let us also define $x_{i+\frac{1}{2}} = (x_i + x_{i+1})/2$ for $i = 0, \cdots, m$, and set $h = \max_{1\le i\le m+1} h_i$.

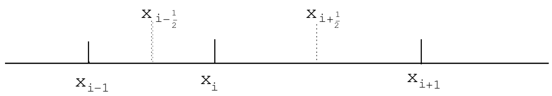

Figure 1: The quasi-uniform line segment mesh (one-dimensional case $d=1$).

Let $\phi_1(x),\cdots,\phi_m(x)$ be a set of piecewise linear basis functions in Ω such that

$$X_h=\text{span}\{\phi_1,\cdots,\phi_m\},$$

with

$$\phi_i(x)=\frac{x-x_{i-1}}{h_i}\,,\quad x\in I_i\,,\quad \phi_i(x)=\frac{x_{i+1}-x}{h_{i+1}}\,,\quad x\in I_{i+1}\,,$$

and

$$\phi_i(x)=0\,,\quad x\in\Omega/(I_i\cup I_{i+1})\,.$$

Similarly, we construct the basis functions at the boundary points:

$$\phi_0(x)=\frac{x_1-x}{h_1}\,,\quad x\in I_1\,,\quad \phi_0(x)=0,\quad x\in\Omega/I_1\,;$$

$$\phi_{m+1}(x)=\frac{x-x_m}{h_{m+1}}\,,\quad x\in I_{m+1}\,,\quad \phi_{m+1}(x)=0,\quad x\in\Omega/I_{m+1}\,.$$

Given the finite element space X_h, the numerical approximation is

$$u_h(x)=\sum_{i=1}^m u^i\phi_i(x)\in X_h$$

with $\nabla u_h(x)=(u^i-u^{i-1})/h_i$ for $x\in I_i$. In the FE solution, $\{u^i\}$ is obtained from

$$\sum_{j=1}^m u^j(\nabla\phi_j,\nabla\phi_i)=\tilde f^{\,i}=\int_{x_{i-1}}^{x_{i+1}}f(x)\phi_i(x)dx\,,$$

or

$$-\frac{1}{h_{i+1}}(u^{i+1}-u^i)+\frac{1}{h_i}(u^i-u^{i-1})=\tilde f^{\,i}, \tag{2.4}$$

where $\int_{x_{i-1}}^{x_{i+1}}\phi_i(x)dx=(h_i+h_{i+1})/2$. In the FV solution, $\{u^i\}$ is obtained from

$$-\left[d_xu_h(x_{i+\frac12})-d_xu_h(x_{i-\frac12})\right]=\bar f^{\,i}=\int_{x_{i-\frac12}}^{x_{i+\frac12}}f(x)dx\,,$$

or

$$-\frac{1}{h_{i+1}}(u^{i+1}-u^i)+\frac{1}{h_i}(u^i-u^{i-1})=\bar{f}^i. \tag{2.5}$$

Finally, in the FD solution $\{u^i\}$ is obtained from

$$-\frac{1}{h_{i+1}}(u^{i+1}-u^i)+\frac{1}{h_i}(u^i-u^{i-1})=f^i=\int_{x_{i-1}}^{x_i} f(x)dx\,. \tag{2.6}$$

We also note that $u^0=u^{m+1}=0$ in all three approximation methods for u_h, and in this section we adopt the notation

$$d_x u(x)=\frac{du(x)}{dx}\,,\quad \Delta u(x)=\frac{d^2u(x)}{dx^2}\,,\quad \delta_x u(x_i)=\frac{u(x_i)-u(x_{i-1})}{h_i}\,.$$

Thus the numerical solution procedure to obtain the approximate solution u_h is to find $U=(u^1,\cdots,u^m)$ such that

$$AU=F, \tag{2.7}$$

where $F=(f_1,\cdots,f_m)^\top$ with $f_i=\tilde{f}^i$, $\bar{f}^i$ and f^i for the FE, FV and FD methods, respectively; and

$$A=\begin{pmatrix} \frac{1}{h_1}+\frac{1}{h_2} & -\frac{1}{h_2} & 0 & \cdots & 0 & 0 \\ -\frac{1}{h_2} & \frac{1}{h_2}+\frac{1}{h_3} & -\frac{1}{h_3} & \cdots & 0 & 0 \\ 0 & -\frac{1}{h_3} & \frac{1}{h_3}+\frac{1}{h_4} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -\frac{1}{h_m} & \frac{1}{h_m}+\frac{1}{h_{m+1}} \end{pmatrix}_{m\times m}.$$

On writing $\nabla u_h|_{[x_{i-1},x_i]}=\delta_x u^i=(u^i-u^{i-1})/h_i$ we can determine the stability of the FE, FV and FD methods as follows.

Lemma 2.1. *If $f\in L^2(\Omega)$, then the numerical solution u_h defined by the three methods (FE, FV and FD) satisfies the following stability criterion and error estimate:*

$$\|\nabla u_h\|_{0,\Omega}\le c\|f\|_{0,\Omega}\,,\quad \|\nabla(u-u_h)\|_{0,\Omega}\le ch\|f\|_{0,\Omega}\,. \tag{2.8}$$

Proof. In the FE and FV solutions for u_h, the inequality (2.8) is well-known [5]. In the case of the FD solution for u_h, taking the R^1-scalar product of Eq. (2.6) with u^i yields

$$\frac{1}{2h_{i+1}}\left[|u^i|^2-|u^{i+1}|^2+|u^i-u^{i+1}|^2\right]+\frac{1}{2h_i}\left[|u^i|^2-|u^{i-1}|^2+|u^i-u^{i-1}|^2\right]\le|f^i||u^i|\,. \tag{2.9}$$

On summing (2.9) from $i=1$ to $i=m$, noting that $u^0=u^{m+1}=0$, and using the modified discrete Poincare inequality [2], we have

$$\sum_{i=1}^{m} h_i|u^i|^2 \le c\sum_{i=1}^{m+1} h_i|\delta_x u^i|^2 , \tag{2.10}$$

and hence the stability criterion

$$h\sum_{i=1}^{m+1} |\delta_x u^i|^2 = \|\nabla u_h\|_{0,\Omega}^2 \le c\|f\|_{0,\Omega}^2 . \tag{2.11}$$

Next, we can deduce the error estimate of the FD solution for u_h. Thus applying the integral operator $\int_{x_{i-1}}^{x_i}\cdots dx$ to Eq. (2.1) and noting that

$$G_x^i = d_x u(x_{i-1}) - \delta_x u(x_i) = \frac{1}{h_i}\int_{x_{i-1}}^{x_i}(x-x_i)\Delta u(x)dx ,$$
$$G_x^{i+1} = d_x u(x_i) - \delta_x u(x_{i+1}) = \frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}}(x-x_{i+1})\Delta u(x)dx ,$$

to obtain

$$-\frac{1}{h_{i+1}}\left(u(x_{i+1})-u(x_i)\right)+\frac{1}{h_i}\left(u(x_i)-u(x_{i-1})\right) = f^i + \left(G_x^{i+1}-G_x^i\right) . \tag{2.12}$$

Subtracting Eq. (2.6) from Eq. (2.12) and writing $e^i=u(x_i)-u^i$, we obtain the error equation

$$-\frac{1}{h_{i+1}}(e^{i+1}-e^i)+\frac{1}{h_i}(e^i-e^{i-1}) = G_x^{i+1}-G_x^i . \tag{2.13}$$

On taking the R^1-scalar product of Eq. (2.13) with e^i,

$$\begin{aligned}&\frac{1}{2h_{i+1}}\left[|e^i|^2-|e^{i+1}|^2+|e^i-e^{i+1}|^2\right]+\frac{1}{2h_i}\left[|e^i|^2-|e^{i-1}|^2+|e^i-e^{i-1}|^2\right]\\ =&(G_x^{i+1}-G_x^i)e^i .\end{aligned} \tag{2.14}$$

Summing Eq. (2.14) from $i=1$ to $i=m$ and noting that $e^0=e^{m+1}=0$, we obtain

$$\begin{aligned}\sum_{i=1}^{m+1} h_i|\delta_x e^i|^2 =& -\sum_{i=1}^{m+1} h_i G_x^i\delta_x e^i \le \left(\sum_{i=1}^{m+1} h_i|G_x^i|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{m+1} h_i|\delta_x e^i|^2\right)^{\frac{1}{2}}\\ \le& h\left(\int_0^1|\Delta u(x)|^2dx\right)^{\frac{1}{2}}\left(\sum_{i=1}^{m+1} h_i|\delta_x e^i|^2\right)^{\frac{1}{2}} ,\end{aligned}$$

so inequality (2.3) yields

$$\|\nabla(I_h u - u_h)\|_{0,\Omega} \le ch\|\Delta u\|_{0,\Omega} \le ch\|f\|_{0,\Omega}\,, \tag{2.15}$$

where $I_h : H_0^1(\Omega) \to X_h$ is an interpolation operator [5] such that

$$I_h u = \sum_{i=1}^{m} u(x_i)\phi_i(x), \quad \|\nabla(u - I_h u)\|_{0,\Omega} \le ch\|\Delta u\|_{0,\Omega}\,. \tag{2.16}$$

From inequalities (2.3) and (2.16), we have

$$\|\nabla(u-u^h)\|_{0,\Omega} \le \|\nabla(u - I_h u)\|_{0,\Omega} + \|\nabla(I_h u - u^h)\|_{0,\Omega} \le ch\|\Delta u\|_{0,\Omega} \le ch\|f\|_{0,\Omega}\,,$$

so inequality (2.11) yields the results (2.8). □

3. FE, FV and FD Methods in Two Dimensions ($d = 2$)

We now construct a quasi-uniform grid for $\Omega = (0,1)\times(0,T)$, where $(0,1)$ is sub-divided as in Section 2 and $(0,T)$ is sub-divided into $y_0 = 0 < y_1 < \cdots < y_N < y_{N+1} = T$. with $\tau_j = y_j - y_{j-1}$ for $j = 0,1,\cdots,N+1$ and $\tau = \max 1 \le j \le N+1\tau_j$. We also define $y_{j+\frac{1}{2}} = (y_{j+1}+y_j)/2$ with $j = 0,\cdots,N$, and assume that $h_i/h_{i-1} \le c_1$ for $i = 1,\cdots,m+1$ and $\tau_j/\tau_{j-1} \le c_2$ for $j = 1,\cdots,N+1$, where $h_0 = h_1$ and $\tau_0 = \tau_1$ are known. For each point $(x_i,y_j) \in (0,1)\times(0,T)$ with $i = 1,\cdots,m$ and $j = 1,\cdots,N$, we construct a macro-element $\tilde{K}_{ij} = (x_{i-1},x_{i+1})\times(y_{j-1},y_{j+1})$ that is divided into six elements $K_{ij}^1,\cdots,K_{ij}^6$, and the dual element $K_{ij}^* = (x_{i-\frac{1}{2}},x_{i+\frac{1}{2}})\times(y_{j-\frac{1}{2}},y_{j+\frac{1}{2}})$ as shown in Fig. 2; and also introduce the basic hat function $\phi_{ij}(x,y)$ such that $\operatorname{supp}\phi_{ij} \subset \tilde{K}_{ij}$ and

$$\phi_{ij}(x,y) = \frac{y-y_{j-1}}{y_j - y_{j-1}} \text{ for } (x,y)\in K_{ij}^1\,, \quad \phi_{ij}(x,y) = \frac{y-y_{j+1}}{y_j-y_{j+1}} \text{ for } (x,y)\in K_{ij}^4\,,$$

$$\phi_{ij}(x,y) = \frac{x-x_{i-1}}{x_i - x_{i-1}} \text{ for } (x,y)\in K_{ij}^6\,, \quad \phi_{ij}(x,y) = \frac{x-x_{i+1}}{x_i-x_{i+1}} \text{ for } (x,y)\in K_{ij}^3\,,$$

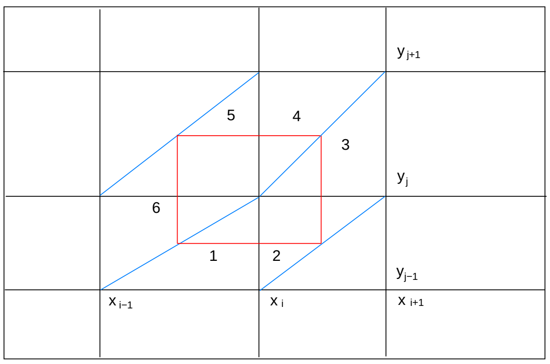

Figure 2: The quasi-uniform rectangular mesh (two-dimensional case $d = 2$).

$$\phi_{ij}(x,y)=1-\frac{x-x_i}{x_{i+1}-x_i}-\frac{y-y_j}{y_{j-1}-y_j}\ \text{for } (x,y)\in K_{ij}^2\,,$$
$$\phi_{ij}(x,y)=1-\frac{x-x_i}{x_{i-1}-x_i}-\frac{y-y_j}{y_{j+1}-y_j}\ \text{for } (x,y)\in K_{ij}^5\,.$$

Similarly, we can construct the basis functions $\phi_{ij}(x,y)$ for $(x_i,y_j)\in\partial\Omega$. In this section, we also adopt the notations

$$\delta_x u^{ij}=\frac{u^{ij}-u^{i-1,j}}{h_i}\,,$$
$$\delta_y u^{ij}=\frac{u^{ij}-u^{i,j-1}}{\tau_j}\,,$$
$$\delta_x u(x_i,y_j)=\frac{u(x_i,y_j)-u(x_{i-1},y_j)}{h_i}\,,$$
$$\delta_y u(x_i,y_j)=\frac{u(x_i,y_j)-u(x_i,y_{j-1})}{\tau_j}\,,$$

and establish the finite element space

$$X_h=\text{span}\{\phi_{11},\cdots,\phi_{m1},\phi_{12},\cdots,\phi_{m2},\cdots,\phi_{1N},\cdots,\phi_{mN}\}\,.$$

Based on the finite element space X_h, the numerical approximation is to find u_h such that $u_h(x)=\sum_{j=1}^{N}\sum_{i=1}^{m}u^{ij}\phi_{ij}(x,y)\in X_h$. In the FE solution, $\{u^{ij}\}$ is obtained from

$$\sum_{l=1}^{N}\sum_{k=1}^{m}u^{kl}(\nabla\phi_{kl},\nabla\phi_{ij})=\tilde{f}^{ij}=\int_{\tilde{K}_{ij}}f(x,y)\phi_{ij}(x,y)\,dx\,dy\,,$$

or

$$-\frac{\tau_{j+1}+\tau_j}{2}(\delta_x u^{i+1,j}-\delta_x u^{ij})-\frac{h_{i+1}+h_i}{2}(\delta_y u^{i,j+1}-\delta_y u^{ij})=\tilde{f}^{i,j}\,. \tag{3.1}$$

In the FV solution, $\{u^{ij}\}$ is obtained from

$$-\int_{\partial K_{ij}^*}\frac{\partial u_h}{\partial n}ds=\bar{f}^{ij}=\int_{K_{ij}^*}f(x,y)dxdy\,,$$

or

$$-\frac{\tau_{j+1}+\tau_j}{2}(\delta_x u^{i+1,j}-\delta_x u^{ij})-\frac{h_{i+1}+h_i}{2}(\delta_y u^{i,j+1}-\delta_y u^{ij})=\bar{f}^{i,j}\,. \tag{3.2}$$

Finally, in the FD solution $\{u^{ij}\}$ is obtained from

$$-\tau_j(\delta_x u^{i+1,j}-\delta_x u^{ij})-h_i(\delta_y u^{i,j+1}-\delta_y u^{ij})=f^{ij}=\int_{K_{ij}}f(x,y)\,dx\,dy\,, \tag{3.3}$$

where $K_{ij}=(x_{i-1},x_i)\times(y_{j-1},y_j)$. In each case, we set $u^{ij}=0$ when $(x_i,y_j)\in\partial\Omega$. Thus the numerical solution procedure is to find $U=(U^1,\cdots,U^N)^\top$ with $U^j=(u^{1j},\cdots,u^{mj})^\top$ for $j=1,\cdots,N$ such that

$$AU=F\,, \tag{3.4}$$

where

$$F=(F_1,F_2,\cdots,F_N)^\top,\quad F_j=(f_{1j},\cdots,f_{mj})^\top$$

with $f_{ij}=\tilde{f}^{ij},\ \bar{f}^{ij}$ and f^{ij} for the FE, FV and FD methods respectively, and A is defined by

$$A=\begin{pmatrix} B_{11} & -B_{12} & 0 & 0 & \cdots & 0 & 0 & 0\\ -B_{21} & B_{22} & -B_{23} & 0 & \cdots & 0 & 0 & 0\\ 0 & -B_{32} & B_{33} & -B_{34} & \cdots & 0 & 0 & 0\\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & 0 & \cdots & -B_{N-1,N-2} & B_{N-1,N-1} & -B_{N-1,N}\\ 0 & 0 & 0 & 0 & \cdots & 0 & -B_{N,N-1} & B_{N,N} \end{pmatrix}_{N\times N}$$

For the FE and FV methods here, we have

$$B_{jj}=\begin{pmatrix} b_{11} & -\frac{\tau_{j+1}+\tau_j}{2h_2} & 0 & 0 & \cdots & 0 & 0 & 0\\ -\frac{\tau_{j+1}+\tau_j}{2h_2} & b_{22} & -\frac{\tau_{j+1}+\tau_j}{2h_3} & 0 & \cdots & 0 & 0 & 0\\ 0 & -\frac{\tau_{j+1}+\tau_j}{2h_3} & b_{33} & -\frac{\tau_{j+1}+\tau_j}{2h_4} & \cdots & 0 & 0 & 0\\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & 0 & \cdots & -\frac{\tau_{j+1}+\tau_j}{2h_{m-1}} & b_{m-1,m-1} & -\frac{\tau_{j+1}+\tau_j}{2h_m}\\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{\tau_{j+1}+\tau_j}{2h_2} & b_{mm} \end{pmatrix}_{m\times m},$$

and

$$B_{j,j+1}=\frac{1}{2\tau_{j+1}}\begin{pmatrix} h_1+h_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & h_2+h_3 & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & h_3+h_4 & 0 & \cdots & 0 & 0 & 0\\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & 0 & \cdots & 0 & h_{m-1}+h_m & 0\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & h_m+h_{m+1} \end{pmatrix}_{m\times m},$$

where

$$B_{j,j+1}=B_{j+1,j}\,,\quad b_{ii}=\frac{\tau_{j+1}+\tau_j}{2}\left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)+\frac{h_i+h_{i+1}}{2}\left(\frac{1}{\tau_{j+1}}+\frac{1}{\tau_j}\right)\,.$$

For the FD method, we have

$$B_{jj} = \begin{pmatrix} b_{11} & -\frac{\tau_j}{h_2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{\tau_j}{h_2} & b_{22} & -\frac{\tau_j}{h_3} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{\tau_j}{h_3} & b_{33} & -\frac{\tau_j}{h_4} & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -\frac{\tau_j}{h_{m-1}} & b_{m-1,m-1} & -\frac{\tau_j}{h_m} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{\tau_j}{h_m} & b_{mm} \end{pmatrix}_{m\times m},$$

and

$$B_{j,j+1} = \frac{1}{\tau_{j+1}} \begin{pmatrix} h_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & h_3 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & h_{m-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & h_m \end{pmatrix}_{m\times m},$$

where

$$B_{j+1,j} = B_{j,j+1}, \quad b_{ii} = \tau_j \left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right) + h_i \left(\frac{1}{\tau_{j+1}} + \frac{1}{\tau_j}\right).$$

If we were to use a uniform mesh, the coefficient matrices of three methods are then the same — viz.

$$A = \begin{pmatrix} B & -\frac{h}{\tau}I & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{h}{\tau}I & B & -\frac{h}{\tau}I & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{h}{\tau}I & B & --\frac{h}{\tau}I & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -\frac{h}{\tau}I & B & -\frac{h}{\tau}I \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{h}{\tau}I & B \end{pmatrix}_{N\times N},$$

$$B = \begin{pmatrix} \frac{2\tau}{h} + \frac{2h}{\tau} & -\frac{\tau}{h} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{\tau}{h} & \frac{2\tau}{h} + \frac{2h}{\tau} & -\frac{\tau}{h} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{\tau}{h} & \frac{2\tau}{h} + \frac{2h}{\tau} & -\frac{\tau}{h} & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -\frac{\tau}{h} & \frac{2\tau}{h} + \frac{2h}{\tau} & -\frac{\tau}{h} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{\tau}{h} & \frac{2\tau}{h} + \frac{2h}{\tau} \end{pmatrix}_{m\times m},$$

where $I = (\delta_{ij})_{m\times m}$ is the relevant unit matrix.

We need the modified discrete Poincare inequality [2]

$$\sum_{j=0}^{N+1} \sum_{i=0}^{m+1} h_i \tau_j |u^{ij}|^2 \le c \sum_{j=1}^{N+1} \sum_{i=1}^{m+1} h_i \tau_j \left[|\delta_x u^{ij}|^2 + |\delta_y u^{ij}|^2 \right], \tag{3.5}$$

to proceed to analyse the stability and error estimate for the FE, FV and FD methods here, and also the following norm equivalent relation.

Lemma 3.1. *For each* $v_h=\sum_{j=1}^{N}\sum_{i=1}^{m} v^{ij}\phi_{ij}(x)\in X_h$,

$$\begin{aligned}&\frac{1}{2}\sum_{j=1}^{N+1}\sum_{i=1}^{m+1}h_i\tau_j\left[|\delta_x v^{ij}|^2+|\delta_y v^{ij}|^2\right]\le\|\nabla v_h\|_{0,\Omega}^2\\ \le & C\sum_{j=1}^{N+1}\sum_{i=1}^{m+1}h_i\tau_j\left[|\delta_x v^{ij}|^2+|\delta_y v^{ij}|^2\right]. \end{aligned}\tag{3.6}$$

where $C=\max\{1,c_1,c_2\}$.

Proof. From the definition of v_h,

$$\begin{aligned} v_h|_{K_{ij}^1}=&v^{i-1,j-1}\phi_{i-1,j-1}(x,y)+v^{i,j-1}\phi_{i,j-1}(x,y)+v^{ij}\phi_{ij}(x,y),\\ v_h|_{K_{ij}^6}=&v^{i-1,j-1}\phi_{i-1,j-1}(x,y)+v^{i-1,j}\phi_{i-1,j}(x,y)+v^{ij}\phi_{ij}(x,y),\end{aligned}$$

which yield

$$\nabla v_h|_{K_{ij}^1}=(\delta_x v^{i,j-1},\delta_y v^{ij})^\top,\quad \nabla v_h|_{K_{ij}^6}=(\delta_x v^{ij},\delta_y v^{i-1,j})^\top.$$

whence

$$\begin{aligned}&\frac{h_i\tau_j}{2}\left[|\delta_x v^{ij}|^2+|\delta_y v^{ij}|^2\right]\le\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}|\nabla v_h|^2\,dx\,dy\\ =&\frac{h_i\tau_j}{2}\left[|\delta_x v^{i,j-1}|^2+|\delta_x v^{ij}|^2+|\delta_y v^{i-1,j}|^2+|\delta_y v^{ij}|^2\right]\\ \le& C\frac{1}{2}\left[|\delta_x v^{i,j-1}|^2h_i\tau_{j-1}+|\delta_x v^{ij}|^2h_i\tau_j+|\delta_y v^{i-1,j}|^2h_{i-1}\tau_j+|\delta_y v^{ij}|^2h_i\tau_j\right].\end{aligned}$$

Summing the above relations for $i=1,\cdots,m+1$ and $j=1,\cdots,N+1$ and noting $v^{ij}=0$ for $(x_i,y_j)\in\partial\Omega$, we obtain inequality (3.6). □

We next consider the stability of the FE, FV and FD methods as follows.

Theorem 3.1. *Assume* u_h *is the solution of* (3.1), (3.2) *or* (3.3), *respectively. Then*

$$\|\nabla u_h\|_{0,\Omega}\le c\|f\|_{0,\Omega}.\tag{3.7}$$

Proof. For the FE and FV methods, it is known that inequality (3.7) holds [5,6]. In the FD case, we take the R^1-scalar product of Eq. (3.3) with u^{ij} to obtain

$$\begin{aligned}&\frac{\tau_j}{2h_{i+1}}\left[|u^{ij}|^2-|u^{i+1,j}|^2+|u^{i+1,j}-u^{ij}|^2\right]+\frac{\tau_j}{2h_i}\left[|u^{ij}|^2-|u^{i-1,j}|^2+|u^{ij}-u^{i-1,j}|^2\right]\\ &+\frac{h_i}{2\tau_{j+1}}\left[|u^{ij}|^2-|u^{i,j+1}|^2+|u^{i,j+1}-u^{ij}|^2\right]+\frac{h_i}{2\tau_j}\left[|u^{ij}|^2-|u^{i,j-1}|^2+|u^{ij}-u^{i,j-1}|^2\right]\\ =&f^{ij}u^{ij}.\end{aligned}\tag{3.8}$$

Summing Eq. (3.8) for $i=1,\cdots,m$ and $j=1,\cdots,N$ and using inequalities (3.5) and (3.6) yields the result (3.7). □

We can also deduce an error estimate of the numerical solution u_h for the FE, FV and FD methods as follows.

Theorem 3.2. *Let $u_h \in X_h$ be the FE solution, the FV solution or the FD solution using Eqs.* (3.1), (3.2) *or* (3.3) *and $f \in L^2(\Omega)$, respectively. Then*

$$\|\nabla u - \nabla u_h\|_{0,\Omega} \le c(h+\tau)\|f\|_{0,\Omega} . \tag{3.9}$$

Proof. For the FE and FV methods, inequality (3.9) holds [5, 6], so it remain to prove this result for the FD method. We apply the integral operator $\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}\cdots dx\,dy$ to Eq. (2.1), so that

$$-\int_{y_{j-1}}^{y_j} \partial_x u(x_i,y)-\partial_x u(x_{i-1},y)dy-\int_{x_{i-1}}^{x_i} \partial_y u(x,y_j)-\partial_y u(x,y_{j-1})\,dx=f^{ij} . \tag{3.10}$$

We note that

$$\begin{aligned}
G_x^i =& \int_{y_{j-1}}^{y_j} \partial_x u(x_{i-1},y)dy-\delta_x u(x_i,y_j)\tau_j \\
=&\frac{1}{h_i}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}(x-x_i)\partial_{xx}u(x,y)\,dx\,dy-\frac{1}{h_i}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}(y-y_{j-1})\partial_{xy}u(x,y)\,dx\,dy , \\
G_x^{i+1} =& \int_{y_{j-1}}^{y_j} \partial_x u(x_i,y)dy-\delta_x u(x_{i+1},y_j)\tau_j \\
=&\frac{1}{h_{i+1}}\int_{y_{j-1}}^{y_j}\int_{x_i}^{x_{i+1}}(x-x_{i+1})\partial_{xx}u(x,y)\,dx\,dy \\
&-\frac{1}{h_{i+1}}\int_{y_{j-1}}^{y_j}\int_{x_i}^{x_{i+1}}(y-y_{j-1})\partial_{xy}u(x,y)\,dx\,dy , \\
G_y^j =& \int_{x_{i-1}}^{x_i} \partial_y u(x,y_{j-1})dx-\delta_y u(x_i,y_j)h_i \\
=&\frac{1}{\tau_j}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}(y-y_j)\partial_{yy}u(x,y)\,dx\,dy-\frac{1}{\tau_j}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}(x-x_{i-1})\partial_{xy}u(x,y)\,dx\,dy ,
\end{aligned} \tag{3.11}$$

and

$$
\begin{aligned}
G_y^{j+1} =& \int_{x_{i-1}}^{x_i} \partial_y u(x,y_j)dx - \delta_y u(x_i,y_{j+1})h_i \\
=& \frac{1}{\tau_{j+1}} \int_{y_j}^{y_{j+1}} \int_{x_{i-1}}^{x_i} (y-y_{j+1})\partial_{yy} u(x,y)\,dx\,dy \\
&- \frac{1}{\tau_{j+1}} \int_{y_j}^{y_{j+1}} \int_{x_{i-1}}^{x_i} (x-x_{i-1})\partial_{xy} u(x,y)\,dx\,dy \ .
\end{aligned}
$$

Combining Eq. (3.10) with the above equations yields

$$
\begin{aligned}
&- \tau_j \left(\delta_x u(x_{i+1},y_j) - \delta_x u(x_i,y_j)\right) - h_i \left(\delta_y u(x_i,y_{j+1}) - \delta_y u(x_i,y_j)\right) \\
=& f^{ij} + G_x^{i+1} - G_x^i + G_y^{j+1} - G_y^j \ .
\end{aligned}
\tag{3.12}
$$

Subtracting Eq. (3.3) from Eq. (3.12) and setting $e^{ij} = u(x_i,y_j) - u^{ij}$, we obtain the error equation

$$
-\tau_j \left(\delta_x e^{i+1,j} - \delta_x e^{ij}\right) - h_i \left(\delta_y e^{i,j+1} - \delta_y e^{ij}\right) = G_x^{i+1} - G_x^i + G_y^{j+1} - G_y^j \ . \tag{3.13}
$$

Then taking the R^1-scalar product of Eq. (3.13) with e^{ij},

$$
\begin{aligned}
&\frac{\tau_j}{2h_{i+1}} \left[|e^{ij}|^2 - |e^{i+1,j}|^2 + |e^{i+1,j} - e^{ij}|^2\right] + \frac{\tau_j}{2h_i} \left(|e^{ij}|^2 - |e^{i-1,j}|^2 + |e^{i-1,j} - e^{ij}|^2\right) \\
&+ \frac{h_i}{2\tau_{j+1}} \left[|e^{ij}|^2 - |e^{i,j+1}|^2 + |e^{i,j+1} - e^{ij}|^2\right] + \frac{h_i}{2\tau_j} \left(|e^{ij}|^2 - |e^{i,j-1}|^2 + |e^{i,j} - e^{i,j-1}|^2\right) \\
=& (G_x^{i+1} - G_x^i)e^{ij} + (G_y^{j+1} - G_y^j)e^{ij} \ .
\end{aligned}
\tag{3.14}
$$

Summing Eq. (3.14) for $i = 1,\cdots,m$ and $j = 1,\cdots,N$, and noting $e^{ij} = 0$ if $(x_i,y_j) \in \partial\Omega$,

$$
\begin{aligned}
&\sum_{j=1}^{N+1}\sum_{i=1}^{m+1} h_i\tau_j \left[|\delta_x e^{ij}|^2 + |\delta_y e^{ij}|^2\right] \\
=& -\sum_{j=1}^{N+1}\sum_{i=1}^{m+1} \left[h_i G_x^i \delta_x e^{ij} + \tau_j G_y^j \delta_y e^{ij}\right] \\
\le& \frac{1}{3}\sum_{j=1}^{N+1}\sum_{i=1}^{m+1} \left(h_i\|\partial_{xx}u\|_{0,K_{ij}} + \tau_j\|\partial_{xy}u\|_{0,K_{ij}}\right)(h_i\tau_j)^{\frac{1}{2}}|\delta_x e^{ij}| \\
&+ \frac{1}{3}\sum_{j=1}^{N+1}\sum_{i=1}^{m+1} \left(\tau_j\|\partial_{yy}u\|_{0,K_{ij}} + h_i\|\partial_{xy}u\|_{0,K_{ij}}\right)(h_i\tau_j)^{\frac{1}{2}}|\delta_y e^{ij}| \\
\le& c(h+\tau)|u|_{2,\Omega} \left[\sum_{j=1}^{N+1}\sum_{i=1}^{m+1}(|\delta_x e^{ij}|^2 + |\delta_y e^{ij}|^2)h_i\tau_j\right]^{\frac{1}{2}} .
\end{aligned}
$$

From (3.7) and the above inequality, the H^1-bound of the error $e_h = I_{h,\tau}u - u^h$ is

$$\|\nabla e_h\|_{0,\Omega} \le c(h+\tau)|u|_{2,\Omega}\,, \tag{3.15}$$

where the interpolation operator $I_{h,\tau}: H^2(\Omega)\cap H_0^1(\Omega) \to X_h$ is defined as

$$I_{h,\tau}u = \sum_{j=1}^{N}\sum_{i=1}^{m} u(x_i, y_j)\phi_{ij}\,.$$

From Ref. [5] we have

$$\|\nabla(I_{h,\tau}u - u)\|_{0,\Omega}^2 \le c(h+\tau)|u|_{2,\Omega}\,, \tag{3.16}$$

so from inequality (2.3) and the inequalities (3.15) and (3.16) we finally obtain the error estimate of the FD solution

$$\begin{aligned}\|\nabla(u-u_h)\|_{0,\Omega} \le& \|\nabla(u - I_{h,\tau}u)\|_{0,\Omega} + \|\nabla e_h\|_{0,\Omega}\\ \le& c(h+\tau)|u|_{2,\Omega} \le c(h+\tau)\|f\|_{0,\Omega}\,,\end{aligned}$$

the stated result. □

4. Numerical Experiments

We now present a test problems in one dimension and another in two dimensions, to illustrate the theoretical results obtained in the previous section.

The first test problem is

$$\begin{aligned}&-u_{xx} = f(x), \qquad x\in(0,1)\,,\\ &u(0) = u(1) = 0\,,\end{aligned}$$

with the exact solution $u(x) = \sin x/\sin 1 - x$ when $f(x) = \sin x/\sin 1$. We use the following two quasi-uniform meshes produced by random numbers in [0,1].

$$T_h = \{0, 0.21396, 0.43866, 0.49831, 0.64349, 0.78886, 1\},$$

$$T_{\widetilde{h}} = \{0, 0.049754, 0.078384, 0.23189, 0.23931, 0.64082, 0.80487, 0.90840, 1\}.$$

We consider three cases:

$$\begin{aligned}&(1.1).\ \text{FD method}: \ f^i = \int_{x_{i-1}}^{x_i} f(x)\,dx\,,\\ &(1.2).\ \text{FV method}: \ \overline{f}^i = \int_{x_{i-\frac12}}^{x_{i+\frac12}} f(x)\,dx\,,\\ &(1.3).\ \text{FE method}: \ \widetilde{f}^i = \int_{x_{i-1}}^{x_{i+1}} f(x)\phi_i(x)\,dx\,,\end{aligned}$$

where $i = 1, 2\cdots, m$. Here we use the Thomas algorithm [12] for the tri-diagonal system (2.7). The numerical results are presented in Tables 1-3, in terms of the H^1-norm and L^2-norm convergence rates.

Table 1: Case (1.1): Numerical results for the first example using the FD method.

Mesh	$\frac{\|u-u_h\|_0}{\|u\|_0}$	Ratio	$\frac{\|u-u_h\|_1}{\|u\|_1}$	Ratio	Mesh	$\frac{\|u-u_h\|_0}{\|u\|_0}$	Ratio	$\frac{\|u-u_h\|_1}{\|u\|_1}$	Ratio
h	1.9611e-1	-	2.6910e-1		$\tilde{h}$	1.7273e-1		3.1680e-1	
$h/2$	9.1155e-2	1.1053	1.3520e-1	0.99305	$\tilde{h}/2$	7.9044e-2	1.1278	1.7005e-1	0.89761
$h/4$	4.3672e-2	1.0616	6.7627e-2	0.99942	$\tilde{h}/4$	3.6965e-2	1.0965	8.7039e-2	0.96623
$h/8$	2.1341e-2	1.0331	3.3803e-2	1.0004	$\tilde{h}/8$	1.7812e-2	1.0533	4.3927e-2	0.98655
$h/16$	1.0544e-2	1.0172	1.6897e-2	1.0004	$\tilde{h}/16$	8.7388e-3	1.0273	2.2054e-2	0.99407
$h/32$	5.2405e-3	1.0086	8.4471e-3	1.0002	$\tilde{h}/32$	4.3281e-3	1.0137	1.1048e-2	0.99726
$h/64$	2.6123e-3	1.0044	4.2231e-3	1.0002	$\tilde{h}/64$	2.1538e-3	1.0068	5.5293e-3	0.99862

Table 2: Case (1.2): Numerical results for the first example using the FV method.

Mesh	$\frac{\|u-u_h\|_0}{\|u\|_0}$	Ratio	$\frac{\|u-u_h\|_1}{\|u\|_1}$	Ratio	Mesh	$\frac{\|u-u_h\|_0}{\|u\|_0}$	Ratio	$\frac{\|u-u_h\|_1}{\|u\|_1}$	Ratio
h	3.9150e-2	-	2.0031e-1		$\tilde{h}$	9.0927e-2		2.5519e-1	
$h/2$	9.8105e-3	1.9966	1.0041e-1	0.99633	$\tilde{h}/2$	2.2906e-2	1.9890	1.2894e-1	0.98487
$h/4$	2.4541e-3	1.9991	5.0238e-2	0.99905	$\tilde{h}/4$	5.7370e-3	1.9974	6.4633e-2	0.99636
$h/8$	6.1360e-4	1.9998	2.5123e-2	0.99977	$\tilde{h}/8$	1.4349e-3	1.9993	3.2337e-2	0.99909
$h/16$	1.5341e-4	1.9999	1.2562e-2	0.99994	$\tilde{h}/16$	3.5877e-4	1.9998	1.6171e-2	0.99978
$h/32$	3.8352e-5	2.0000	6.2810e-3	1.0000	$\tilde{h}/32$	8.9694e-5	2.0000	8.0859e-3	0.99993
$h/64$	9.5880e-6	2.0000	3.1405e-3	1.0000	$\tilde{h}/64$	2.2424e-5	2.0000	4.0430e-3	0.99998

Table 3: Case (1.3): Numerical results for the first example using the FE method.

Mesh	$\frac{\|u-u_h\|_0}{\|u\|_0}$	Ratio	$\frac{\|u-u_h\|_1}{\|u\|_1}$	Ratio	Mesh	$\frac{\|u-u_h\|_0}{\|u\|_0}$	Ratio	$\frac{\|u-u_h\|_1}{\|u\|_1}$	Ratio
h	4.0574e-2	-	2.0026e-1		$\tilde{h}$	9.3691e-2		2.5445e-1	
$h/2$	1.0183e-2	1.9944	1.0041e-1	0.99597	$\tilde{h}/2$	2.3861e-2	1.9733	1.2885e-1	0.98169
$h/4$	2.5483e-3	1.9986	5.0237e-2	0.99908	$\tilde{h}/4$	5.9920e-3	1.9935	6.4622e-2	0.99560
$h/8$	6.3724e-4	1.9996	2.5123e-2	0.99974	$\tilde{h}/8$	1.4997e-3	1.9984	3.2336e-2	0.99888
$h/16$	1.5932e-4	1.9999	1.2562e-2	0.99994	$\tilde{h}/16$	3.7502e-4	1.9996	1.6171e-2	0.99973
$h/32$	3.9830e-5	2.0000	6.2810e-3	1.0000	$\tilde{h}/32$	9.3762e-5	1.9999	8.0859e-3	0.99993
$h/64$	9.9576e-6	2.0000	3.1405e-3	1.0000	$\tilde{h}/64$	2.3441e-5	2.0000	4.0430e-3	0.99998

The second test problem is

$$-u_{xx} - u_{yy} = f(x,y)\,, \quad (x,y) \in (0,1)\times(0,1)\,,$$

$$u|_{\partial\Omega} = 0\,,$$

where the exact solution is $u(x,y) = \sin(2\pi x)\sin(2\pi y)(x^3 - y^4 + x^2y^3)$ and the right-hand side function is generated by $f = -\Delta u$. We again consider the three cases

$$\text{(2.1). FD method} : f^{ij} = \int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i} f(x,y)\,dx\,dy\,,$$
$$\text{(2.2). FV method} : \overline{f}^{ij} = \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x,y)\,dx\,dy\,,$$
$$\text{(2.3). FE method} : \widetilde{f}^{ij} = \int_{D_{ij}} f(x,y)\phi_{ij}(x,y)\,dx\,dy\,,$$

where $D_{ij} = \sum_{k=1}^{6} K_{ij}^{k}$ and $i = 1,2\cdots,m; j = 1,2\cdots,N$. We use the two quasi-uniform meshes produced by random numbers in [0,1].

$$T_h = \{0, 0.37948, 0.50281, 0.68128, 0.83180, 1\} \times \{0, 0.30462, 0.42889, 0.70947, 1\},$$

$$T_{\widetilde{h}} = \{0, 0.17296, 0.27145, 0.52259, 0.88014, 0.97975, 1\}\times$$
$$\{0, 0.011757, 0.13652, 0.25233, 0.73731, 0.87574, 0.89390, 1\}.$$

Table 4: Case (2.1): Numerical results for the second example using the FD method.

Mesh	$\frac{\|u-u_h\|_0}{\|u\|_0}$	Ratio	$\frac{\|u-u_h\|_1}{\|u\|_1}$	Ratio	Mesh	$\frac{\|u-u_h\|_0}{\|u\|_0}$	Ratio	$\frac{\|u-u_h\|_1}{\|u\|_1}$	Ratio
h	9.1995e-1	-	8.8531e-1		$\widetilde{h}$	6.9511e-1		8.4733e-1	
$h/2$	5.1646e-1	0.83290	5.8918e-1	0.58747	$\widetilde{h}/2$	6.7081e-1	0.35586	5.4316e-1	0.33702
$h/4$	2.4924e-1	1.0511	3.1940e-1	0.88334	$\widetilde{h}/4$	3.1004e-1	0.80892	3.8602e-1	0.79723
$h/8$	1.1915e-1	1.0648	1.6306e-1	0.96996	$\widetilde{h}/8$	1.6227e-1	0.93406	2.0163e-1	0.93697
$h/16$	5.7883e-2	1.0416	8.1883e-2	0.99377	$\widetilde{h}/16$	8.2541e-2	0.97521	1.0230e-1	0.97890
$h/32$	2.8489e-2	1.0227	4.0961e-2	0.99931	$\widetilde{h}/32$	4.1565e-2	0.98974	5.1430e-2	0.99212
$h/64$	1.4128e-2	1.0118	2.0477e-2	1.0002	$\widetilde{h}/64$	2.0849e-2	0.99539	2.5772e-2	0.99681

Table 5: Case (2.2): Numerical results for the second example using the FV method.

Mesh	$\frac{\|u-u_h\|_0}{\|u\|_0}$	Ratio	$\frac{\|u-u_h\|_1}{\|u\|_1}$	Ratio	Mesh	$\frac{\|u-u_h\|_0}{\|u\|_0}$	Ratio	$\frac{\|u-u_h\|_1}{\|u\|_1}$	Ratio
h	6.1573e-1	-	7.8709e-1		$\widetilde{h}$	4.1946e-1		7.1806e-1	
$h/2$	1.8990e-1	1.6971	4.5302e-1	0.79695	$\widetilde{h}/2$	2.2947e-1	0.87023	4.6248e-1	0.63471
$h/4$	5.0682e-2	1.9057	2.3515e-1	0.94599	$\widetilde{h}/4$	6.2350e-2	1.8798	2.4407e-1	0.92210
$h/8$	1.2898e-2	1.9743	1.1867e-1	0.98663	$\widetilde{h}/8$	1.5931e-2	1.9686	1.2375e-1	0.97987
$h/16$	3.2395e-3	1.9933	5.9441e-2	0.99742	$\widetilde{h}/16$	4.0046e-3	1.9921	6.2086e-2	0.99509
$h/32$	8.1082e-4	1.9983	2.9724e-2	0.99983	$\widetilde{h}/32$	1.0025e-3	1.9981	3.1069e-2	0.99879
$h/64$	2.0277e-4	1.9995	1.4860e-2	1.0002	$\widetilde{h}/64$	2.5071e-4	1.9995	1.5538e-2	0.99967

From this procedure, we get the linear systems (3.4) where the coefficient matrices are positive definite and symmetric. We use the standard conjugate gradient method [14] as an iterative solver with initial guess ($x_0 = 0.0$), and the stopping criterion that the norm of

Table 6: Case (2.3): Numerical results for the second example using the FE method.

Mesh	$\frac{\|u-u_h\|_0}{\|u\|_0}$	Ratio	$\frac{\|u-u_h\|_1}{\|u\|_1}$	Ratio	Mesh	$\frac{\|u-u_h\|_0}{\|u\|_0}$	Ratio	$\frac{\|u-u_h\|_1}{\|u\|_1}$	Ratio
h	6.8008e-1	-	7.7041e-1		$\widetilde{h}$	4.5829e-1		6.7125e-1	
$h/2$	2.2124e-1	1.6201	4.4972e-1	0.77660	$\widetilde{h}/2$	2.6407e-1	0.79534	4.5342e-1	0.56600
$h/4$	6.0485e-2	1.8710	2.3465e-1	0.93852	$\widetilde{h}/4$	7.7233e-2	1.7736	2.4241e-1	0.90340
$h/8$	1.5516e-2	1.9628	1.1860e-1	0.98441	$\widetilde{h}/8$	2.0210e-2	1.9341	1.2350e-1	0.97294
$h/16$	3.9055e-3	1.9902	5.9433e-2	0.99677	$\widetilde{h}/16$	5.1147e-3	1.9823	6.2053e-2	0.99294
$h/32$	9.7806e-4	1.9975	2.9723e-2	0.99968	$\widetilde{h}/32$	1.2827e-3	1.9955	3.1065e-2	0.99821
$h/64$	2.4463e-4	1.9993	1.4860e-2	1.0001	$\widetilde{h}/64$	3.2095e-4	1.9988	1.5537e-2	0.99958

the residual vectors is less than 1.0×10^{-14}. The numerical results are presented in Tables 4-6, in terms of the H^1-norm and L^2-norm convergence rates.

We observe from these tables that the convergence rates of the three methods in the H^1-norm are optimal on the quasi-uniform meshes, which confirms our theoretical analysis. As expected, the errors become smaller as the meshes are refined. Moreover, we see from the numerical results that the convergence rates of both the FE and FV methods in the L^2-norm are of order two, but the convergence rate of the FD method in the L^2-norm is only of order one on the quasi-uniform meshes.

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