On Optimal Cash Management under a Stochastic Volatility Model

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Abstract. We discuss a mathematical model for optimal cash management. A firm wishes to manage cash to meet demands for daily operations, and to maximize terminal wealth via bank deposits and stock investments that pay dividends and have uncertain capital gains. A Stochastic Volatility (SV) model is adopted for the capital gains rate of a stock, providing a more realistic way to describe its price dynamics. The cash management problem is formulated as a stochastic optimal control problem, and solved numerically using dynamic programming. We analyze the implications of the heteroscedasticity described by the SV model for evaluating risk, by comparing the terminal wealth arising from the SV model to that obtained from a Constant Volatility (CV) model.

1. Introduction

We investigate a cash management problem where the cash of a firm is divided into two parts — viz. cash required for daily operations, and cash invested in securities such as bank accounts and stocks. On the one hand, the managers need to control the cash balance

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to meet the continual cash demand of the firm, but on the other hand invest in securities in order to maximize the terminal value of total assets. This cash management problem also arises in financial planning for an individual, to provide money for living expenses and yet maximize personal total wealth at some terminal time.

Sethi and Thompson [10] presented a deterministic model allowing for time-varying cash demand, and solved the cash management problem using maximum principles in both discrete-time and continuous-time frameworks. Bensoussan *et al.* [1] extended the Sethi-Thompson model to allow capital gains on stock, and also presented a stochastic model where the rate of growth in the price of the stock is considered random. Using the stochastic maximum principle, they obtained an analytic solution for the optimal cash management problem.

In the stochastic model developed by Bensoussan *et al.* [1], the volatility of the stock dividends is a deterministic function of time, and we introduce a Stochastic Volatility (SV) model for the rate of growth in the price of a stock. In stochastic volatility models, the volatility changes randomly over time according to some stochastic differential equations or some discrete-time random processes, which have been widely used in financial economics and mathematical finance.

Many empirical studies have shown that stochastic volatility models often provide a more realistic description than Constant Volatility (CV) models — e.g. see Taylor [13, 14]. Indeed, many theories in mathematical finance are built on continuous-time models, into which stochastic volatility models tend to fit naturally for a wide array of applications — e.g. the pricing of currencies or options and other derivatives, or in modeling the term structure of interest rates. For example, Pillay & O' Hara [9] considered the pricing of European options when the underlying asset follows a mean reverting log-normal process with stochastic volatility, and Zvan *et al.* [18] developed penalty methods for American options with stochastic volatility. Some excellent surveys on stochastic volatility models include Ghysels *et al.* [4] and Shephard [11], and the literature is already vast and rapidly growing. Other references on continuous-time stochastic volatility models include Chen [3], Hull & White [6], Kilin [7], and Wiggins [15].

In this article, we analyze the implications of the heteroscedastic effect in a continuous-time Stochastic Volatility model (SV) for evaluating risk in the cash management problem, by comparing the terminal wealth processes derived from the SV and CV models. We formulate the optimal cash management problem as a stochastic optimal control problem and solve it numerically using dynamic programming, although there are other approaches — e.g. a genetic algorithm as in Ref. [17], to solve the asset allocation problem. In Section 2, we describe the cash management problem and derive the Hamilton-Jacobi-Bellman (HJB) second-order nonlinear partial differential equation governing the value function of the optimization problem. The successive approximation algorithm introduced by Chang & Krishna [2] we employ to solve the HJB equation is discussed in Section 3, where our main results and some sensitivity analyses are also presented. Brief Concluding Remarks are made in Section 4.

2. The Optimal Cash Management Problem

2.1. Preliminary

Consider an n-dimensional stochastic process $\mathbf{x} := \{\mathbf{x}_t\}_{t \geq 0}$ is governed by the stochastic differential equation

$$d\mathbf{x}_t = f(t, \mathbf{x}_t, \mathbf{u}_t)dt + g(t, \mathbf{x}_t, \mathbf{u}_t)dW_t ,$$

where $W := \{W_t\}_{t \geq 0}$ denotes a k-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geq 0}, \mathscr{P})$, and the filtration $\{\mathscr{F}_t\}_{t \geq 0}$ satisfies some standard conditions — viz. right-continuity and $\mathscr{P}$-completeness. The process $\mathbf{x}$ is adapted to the filtration $\{\mathscr{F}_t\}_{t \geq 0}$, and the vector $\mathbf{u}$ represents the control variable with values in a given compact convex set $U \subset \mathbb{R}^m$. We suppose the state process $\mathbf{x}$ takes values in $G \subset \mathbb{R}^n$, and the drift term $f(t, \mathbf{x}, \mathbf{u})$ and the diffusion $g(t, \mathbf{x}, \mathbf{u})$ are given functions

$$f : [0, T] \times G \times U \to \mathbb{R}^n ,$$
$$g : [0, T] \times G \times U \to \mathbb{R}^{n \times k} ,$$

which satisfy Lipschitz conditions and linear growth conditions such that the stochastic differential equation for the state process $\mathbf{x}$ has a unique strong solution. The performance functional is then given by

$$V(t, \mathbf{x}, \mathbf{u}) = E_{t,x}\left\{ \int_t^T L(s, \mathbf{x}_s, \mathbf{u}_s)ds + K(T, \mathbf{x}(T)) \right\}$$

where $E_{t,x}$ denotes the conditional expectation given $\mathbf{x}_t = \mathbf{x}$ under $\mathscr{P}$, and L and K are scalar functions — viz.

$$L : [0, T] \times G \times U \to \mathbb{R} ,$$
$$K : [0, T] \times G \to \mathbb{R} .$$

Suppose that

$$E\left\{ \int_0^T |L(t, \mathbf{x}_t, \mathbf{u}_t)|dt + |K(T, \mathbf{x}(T))| \right\} < \infty ,$$

where E is the expectation under $\mathscr{P}$.

A control $\mathbf{u}$ is said to be admissible if it satisfies the following conditions:

1. $\mathbf{u}$ is $\{\mathscr{F}_t\}_{t \geq 0}$-progressively measurable;

2. for each $t \geq 0$, $\mathbf{u}_t \in U$; and

3. the stochastic differential equation for the controlled state process $\mathbf{x}$ has a unique strong solution.

Our objective is to find the admissible control $\mathbf{u}$ that maximizes the value of the performance functional $V(t, \mathbf{x}, \mathbf{u})$, leading to the value function

$$v(t, \mathbf{x}) = \max_{\mathbf{u}(t,\mathbf{x}) \in U} V(t, \mathbf{x}, \mathbf{u}) .$$

A necessary condition of the value function is that, under some suitable differentiability conditions, $v(t, \mathbf{x})$ is the solution of the following Hamilton-Jacobi-Bellman (HJB) equation (e.g. see Peyrl *et al.* [8], Yong & Zhou [16]):

$$v_t + \max_{\mathbf{u} \in U} \left\{ L(t, \mathbf{x}, \mathbf{u}) + \mathscr{A}(t, \mathbf{x}, \mathbf{u})v \right\} = 0 , \quad (t, \mathbf{x}) \in Q , \qquad (2.1)$$

with the boundary condition

$$v(t, \mathbf{x}) = K(t, \mathbf{x}) , \quad (t, \mathbf{x}) \in \partial Q ,$$

where $Q := (0, T) \times G$ represents the solvency region. Eq. (2.1) involves the second-order differential operator

$$\mathscr{A}(t, \mathbf{x}, \mathbf{u}) = \frac{1}{2} \sum_{i,j=1}^{n} \sigma_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} ,$$

and the symmetric matrix

$$\sigma(t, \mathbf{x}, \mathbf{u}) = g(t, \mathbf{x}, \mathbf{u}) g^T(t, \mathbf{x}, \mathbf{u}) .$$

2.2. Problem formulation

We consider an optimal cash management problem, by extending the model in Ref. [1]. We assume that the firm invests its cash in a bank account and a stock in a portfolio of value $w(t)$ at time t, and the proportion of wealth invested in the stock at time t is $u(t)$. The interest rate earned in the bank account is r_1, and the return from the stock at time t has two components — viz, the cash dividend rate r_2 and the capital gain rate $R(t)$. Mathematically, the dynamic of the capital gain rate $R(t)$ is assumed to be governed by the stochastic process

$$dR(t) = (f + \beta_1 R(t))dt + \sigma(t)dW_1(t) , \quad R(0) = R_0 ,$$

with the volatility of $R(t)$ modeled by

$$d\sigma(t) = \alpha\sigma(t)dt + \beta\sigma(t)dW_2(t) , \quad \sigma(0) = \sigma_0$$

— i.e. the capital gain rate $R(t)$ is described by a mean-reverting stochastic volatility model, and the stochastic volatility is a log-normal process.

Suppose the firm has a demand rate $d(t)$ for cash at time t, and that the demand rate $d(t)$ is normally distributed with mean 0 and variance 0.2. The control variable is

the proportion of wealth invested in the stock market at time t. It is not unreasonable to assume that $u(t) \in [0,1]$ for any $t \in [1,T]$, so that short-selling is not allowed. The wealth dynamics for this cash management problem is then given by

$$dw(t) = w(t)u(t)r_2 dt + w(t)(1-u(t))r_1 dt + w(t)u(t)R(t)dt - d(t)w(t)dt \,,$$
$$w(0) = w_0 \,,$$

and the objective of the firm is to maximize the expectation of the total holdings at the terminal time T. Thus the portfolio optimization problem is given by

$$\max_{u(t)} \{E[w(T)]\} \,, \tag{2.2}$$

subject to

$$\begin{cases} dw(t) = w(t)u(t)r_2 dt + w(t)(1-u(t))r_1 dt \\ \qquad + w(t)u(t)R(t)dt - d(t)w(t)dt \,, \quad w(0) = w_0 \,, \\ dR(t) = (f + \beta_1 R(t))dt + \sigma(t)dW_1(t) \,, \quad R(0) = R_0 \,, \\ d\sigma(t) = \alpha\sigma(t)dt + \beta\sigma(t)dW_2(t) \,, \quad \sigma(0) = \sigma_0 \,. \end{cases}$$

We assume that the two Brownian motions are correlated — i.e.

$$dW_1 dW_2 = \rho dt.$$

According to Eq. (2.1), the HJB equation for the portfolio problem (2.2) is then given by

$$v_t + \max_{0 \le u(t) \le 1} \left\{ w(t)\Big(u(t)r_2 + (1-u(t))r_1 + u(t)R(t) - d(t)\Big)v_w \right.$$
$$+ (f + \beta_1 R(t))v_R + \Big(\alpha\sigma(t)\Big)v_\sigma$$
$$\left. + \frac{1}{2}\Big(v_{RR}\sigma^2(t) + v_{\sigma\sigma}\beta^2\sigma^2(t)\Big) + \rho v_{R\sigma}\beta\sigma^2(t) \right\} = 0 \,, \tag{2.3}$$

with terminal condition $v(T, \cdot) = w(T)$. We assume that $v = w(t)h(t,R,\sigma)$ to simplify our problem as

$$h_t + \max_{0 \le u(t) \le 1} \left\{ \Big(u(t)r_2 + (1-u(t))r_1 + u(t)R(t) - d(t)\Big)h \right.$$
$$+ (f + \beta_1 R(t))h_R + \Big(\alpha\sigma(t)\Big)h_\sigma$$
$$\left. + \frac{1}{2}\Big(h_{RR}\sigma^2(t) + h_{\sigma\sigma}\beta^2\sigma^2(t)\Big) + \rho h_{R\sigma}\beta\sigma^2(t) \right\} = 0 \,, \tag{2.4}$$

with the terminal condition $h(T, \cdot) = 1$.

3. Numerical Experiments

In many mathematical finance problems, the risk is described by a constant volatility parameter, but following many empirical studies it has since been proposed that volatilities should be modeled by stochastic processes. In this section, we analyze the implications of the heteroscedastic effect described by the SV model for evaluating risk in the cash management problem, by comparing the optimal cash management results arising from the SV model with that from the CV model. We also conduct numerical experiments to provide sensitivity analysis.

3.1. The successive approximation algorithm

As previously mentioned, a necessary condition for an optimal solution of a stochastic control problem is the HJB second-order nonlinear partial differential equation. Analytical solutions of the HJB-PDE can be obtained for some special cases with simple state equations, but more generally we can apply a successive approximation algorithm for a numerical solution via two subproblems [2]:

(1) Numerical solution of the PDE, and

(2) Optimization of the nonlinear function over u.

The finite difference scheme introduced by Strikwerda [12] is employed to solve the HJB-PDE, which in our model (2.4) has a terminal condition $h(T, \cdot) = 1$ rather than an initial condition, and is therefore integrated backwards in time. Specifically, we employ an explicit scheme and use backward differences to approximate first-order derivatives, but central space differences for second-order and mixed derivatives. For the optimization subproblem, an explicit expression for the optimal control law can be obtained according to Eq. (2.3). The steps in the iterative algorithm are as follows:

Step I: Suppose $k = 1$. Choose an arbitrary initial control law $u^0 \in [0, 1]$, and proceed to solve the PDE — i.e. find the solution $h^0(t, R, \sigma)$ of

$$
h_t^0 + \left\{ \left(u^0 r_2 + (1 - u^0) r_1 + u^0 R(t) - d(t) \right) h^0 + (f + \beta_1 R(t)) h_R^0 + \left(\alpha \sigma(t) \right) h_\sigma^0 \right.
$$
$$
\left. + \frac{1}{2} \left(h_{RR}^0 \sigma^2(t) + h_{\sigma\sigma}^0 \beta^2 \sigma^2(t) \right) + \rho h_{R\sigma}^0 \beta \sigma^2(t) \right\} = 0 ,
$$

with $h^0(T, \cdot) = 1$. (According to the proof of the convergence of this iterative algorithm in Ref. [5], an arbitrary choice of the initial control law does not affect the convergence property.)

Step II: Compute the succeeding control law u^k — i.e. solve the optimization problem reduced to (2.4) via

$$
\max_{u^k \in [0,1]} \left[u^k (r_2 + R - r_1) h^{k-1} \right] .
$$

The decision rule arising from the problem is of "bang-bang" type — i.e.

$$\mu^k = \begin{cases} 1, & \text{if } r_2 + R - r_1 > 0 \\ 0, & \text{if } r_2 + R - r_1 \leq 0 \end{cases} ; \tag{3.1}$$

and the PDE for the fixed control law u^k becomes

$$h_t^k + \left\{ \left(u^k r_2 + (1 - u^k) r_1 + u^k R(t) - d(t) \right) h^k + (f + \beta_1 R(t)) h_R^k + \left(\alpha \sigma(t) \right) h_\sigma^k \right.$$
$$\left. + \frac{1}{2} \left(h_{RR}^k \sigma^2(t) + h_{\sigma\sigma}^k \beta^2 \sigma^2(t) \right) + \rho h_{R\sigma}^k \beta \sigma^2(t) \right\} = 0 ,$$

with $h^k(T, \cdot) = 1$.

Step III: Repeat Step II with $k = k + 1$ until $\|h^{k-1} - h^k\| < \epsilon$, for ϵ sufficiently small.

The implementation was in MATLAB, and we found the optimal solutions of the problem can be obtained in about 10 iterations of this successive approximation algorithm.

3.2. Optimal choice under the SV and CV models

Numerical experiments were conducted to compare the results using the SV model with that obtained from the CV model. We considered some hypothetical values for the model parameters — viz. $T = 10$ years, $r_1 = 0.024$, $r_3 = 0.01$, $f = 0.12$, $\beta_1 = 0.96$, $\alpha = -0.85$ and $\beta = 0.3$. In this subsection, we compare the optimal cash decision arising from the two models.

Table 1 shows the optimal choices predicted by the SV and CV models, respectively. According to Eq. (3.1), the decision rule is of "bang-bang" type. Fig. 1 depicts the wealth dynamic results for the firm for the times 1 to 10. We consider three investment methods — viz. (1) the optimal investment according to Eq. (3.1); (2) investment of all of the money in the stock; and (3) all of the money deposited into the bank account. As illustrated in Fig. 1, if the increasing rate of the stock wealth is greater than the bank account wealth (e.g. at time 5), then the manager should invest all of the cash in the stock.

However, if the decreasing rate of the stock wealth is greater than that of bank account wealth (e.g. at time 7, it is optimal for the manager to transfer the cash to the bank account. As shown in Fig. 1, the wealth dynamics plots derived from the two models are almost the same, except at time 9 when the earning from the stock is greater than that from the bank account under the SV model. Meanwhile, the stock wealth under the CV model decreases — so as indicated in Table 1, the manager should transfer all of the cash into the bank

Table 1: Optimal choice from CV model and SV model.

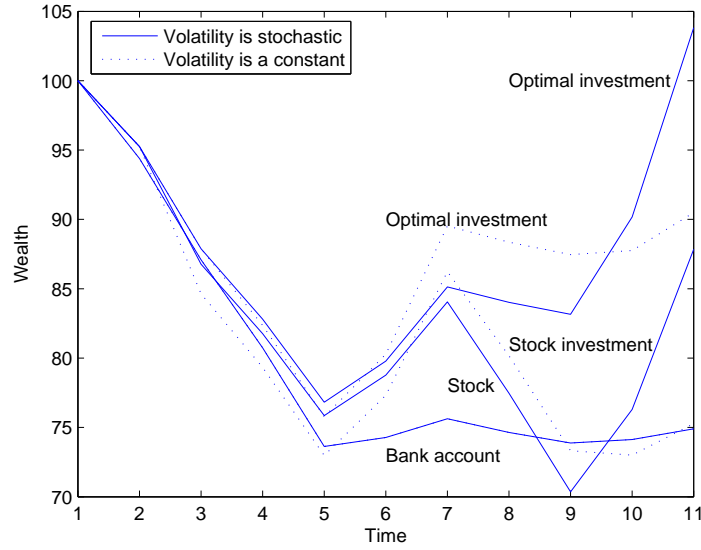| Time	1	2	3	4	5	6	7	8	9	10
Optimal choice when using the CV model	1	0	1	1	1	1	0	0	0	1
Optimal choice when using the SV model	1	0	1	1	1	1	0	0	1	1

Figure 1: Portfolio value from $t = 1$ to $t = 10$.

account according to the CV model, but take no action according to the SV model. Fig. 1 shows the heteroscedastic effect on terminal wealth for the cash management problem, where we observe that the wealth under the SV model varies more dramatically than that from CV model, and the SV model improves the terminal wealth significantly compared to the CV model. This makes intuitive sense, as the stochastic volatility model introduces an additional source of uncertainty. The density function for the capital gains rate R from the SV model has a heavier tail than under the CV model, and heavy tails imply that the risk of extreme movements is high. When the stock price varies dramatically, investors have more chance to make profits according to our optimal policy.

3.3. Sensitivity to the parameter f

In this and the following subsection, we report on numerical experiments to provide some sensitivity analysis for the optimal cash management. Our sensitivity analysis methodology is to vary one parameter at a time while keeping other parameters fixed, to evaluate the influence of the varied parameter on the optimal investment derived from the CV and SV models.

Firstly, we focus on the effect of f, varied from 0.08 to 0.17. Fig. 2 depicts the terminal wealth processes for both the CV and SV models against the parameter f. The SV model improves the terminal wealth significantly, compared to the CV model in Fig. 2. Again this make intuitive sense, as the distribution of the capital gains rate R from the SV model has a heavier tail than under the CV model, and the associated more extreme events result in a greater chance for the manager to make larger profits. By comparing the first sub-figure with the other sub-figures in Fig. 2, one can see that in both models a lower value of β_1
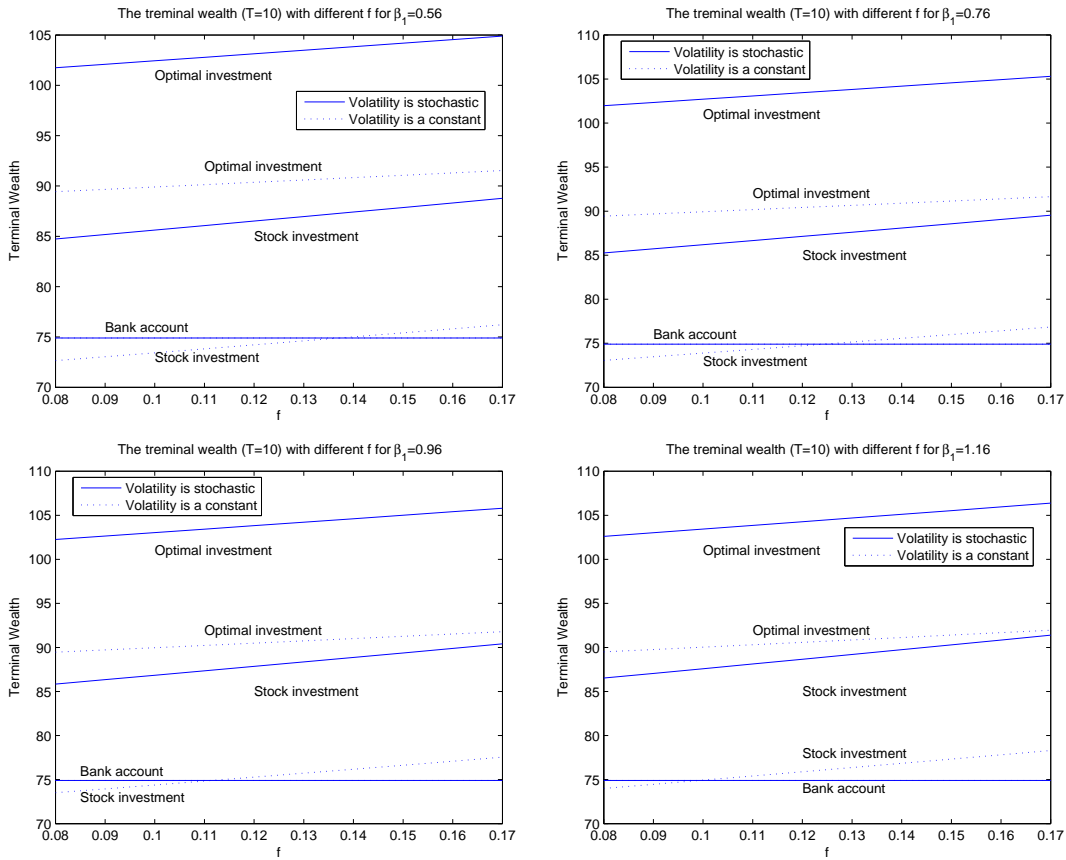
Figure 2: The terminal wealth $(T = 10)$ with different f.

gives a lower terminal wealth and a higher β_1 gives a higher terminal wealth, while the terminal wealth increases with f. These results are consistent with the numerical findings presented in Fig. 3.

From all four sub-figures in Fig. 2, we see that the terminal wealth derived from the SV model increases more rapidly than under the CV model as the value of f increases. This can be explained by the stochastic volatility introducing an additional source of uncertainty, when changes in the value of the parameter f may have a more significant impact on terminal wealth. In brief, the SV model is more sensitive to the parameter f than the CV model. Further, if we change the value of β_1 while the parameter f increases from 0.08 to 0.17, the influence on terminal wealth under the SV model is greater than that from the CV model — cf. also Fig. 3.

3.4. Sensitivity to the parameter β_1

The influence of the parameter β_1 on the optimal investment was also considered, where β_1 varies from 0.56 to 2.36. Fig. 3 depicts the predicted terminal wealth for both
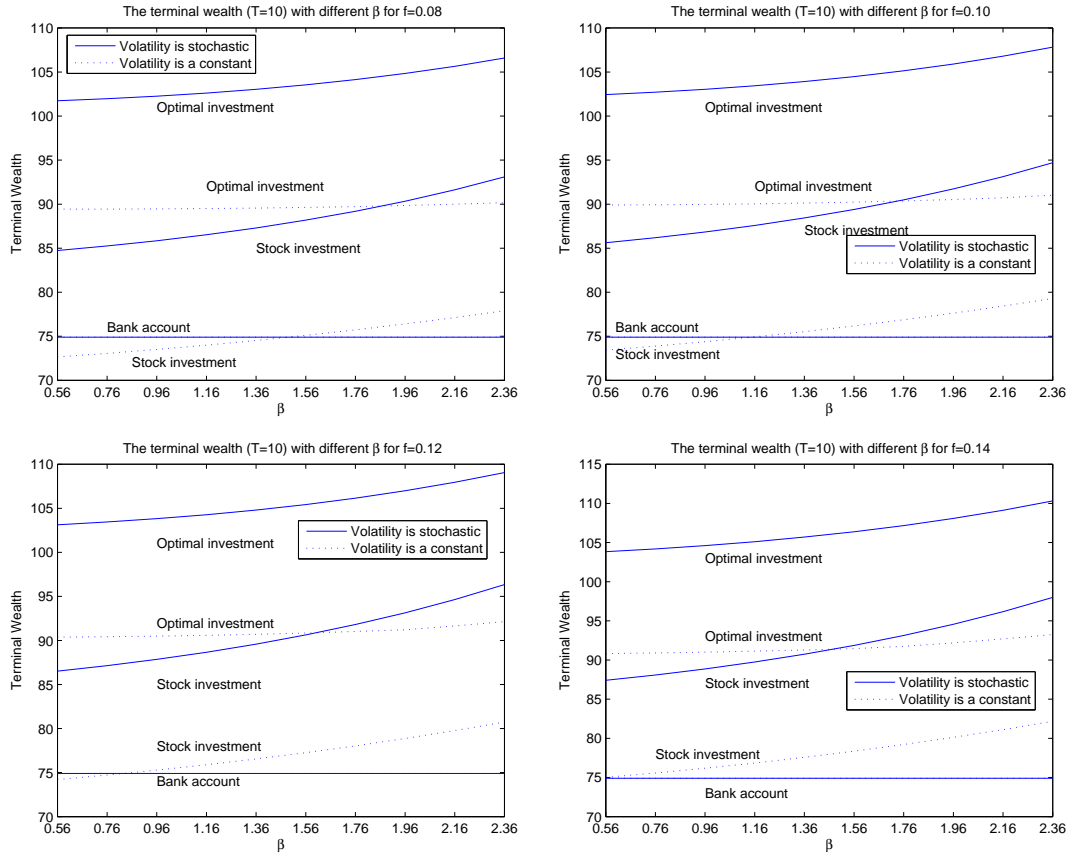
Figure 3: The terminal wealth $(T = 10)$ with different β_1.

the CV and SV models against β_1, where again the terminal wealth derived from the SV model is greater than that from the CV model, and also increases more rapidly with β_1. When f varies from 0.08 to 0.14, the curves for the terminal values move upward for both the SV and CV models — but more significantly for the SV model, consistent with the numerical results shown in Fig. 2. In brief, we find that the SV model is more sensitive to the parameter β_1 than the CV model.

4. Concluding Remarks

A model for optimal cash management in a stochastic volatility model was presented. The cash management problem involves two types of asset, a risky stock and a bank account deposit. The capital gains rate of the stock in our approach follows a stochastic volatility (SV) model. Using dynamic programming, terminal wealth was derived from the SV and a CV model and the implications of the heteroscedastic effect described by the SV model for evaluating risk in the cash management problem analyzed. Numerical experiments to provide sensitivity analysis for the optimal cash management problem were also

conducted. Further research could consider a more reasonable model for the cash demand, or transaction costs in the market, or a constraint that the bank account must maintain a nonnegative balance. Theory for the convergence rate of our proposed iterative scheme might also be developed.

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