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On Optimal Cash Management under a Stochastic Volatility Model

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Abstract. We discuss a mathematical model for optimal cash management. A firm wishes to manage cash to meet demands for daily operations, and to maximize terminal wealth via bank deposits and stock investments that pay dividends and have uncertain capital gains. A Stochastic Volatility (SV) model is adopted for the capital gains rate of a stock, providing a more realistic way to describe its price dynamics. The cash management problem is formulated as a stochastic optimal control problem, and solved numerically using dynamic programming. We analyze the implications of the heteroscedasticity described by the SV model for evaluating risk, by comparing the terminal wealth arising from the SV model to that obtained from a Constant Volatility (CV) model.

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Key words: Optimal cash management, stochastic volatility, dynamic programming, HJB equations.

1. Introduction

We investigate a cash management problem where the cash of a firm is divided into two parts — viz. cash required for daily operations, and cash invested in securities such as bank accounts and stocks. On the one hand, the managers need to control the cash balance

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to meet the continual cash demand of the firm, but on the other hand invest in securities in order to maximize the terminal value of total assets. This cash management problem also arises in financial planning for an individual, to provide money for living expenses and yet maximize personal total wealth at some terminal time.

Sethi and Thompson [10] presented a deterministic model allowing for time-varying cash demand, and solved the cash management problem using maximum principles in both discrete-time and continuous-time frameworks. Bensoussan *et al.* [1] extended the Sethi-Thompson model to allow capital gains on stock, and also presented a stochastic model where the rate of growth in the price of the stock is considered random. Using the stochastic maximum principle, they obtained an analytic solution for the optimal cash management problem.

In the stochastic model developed by Bensoussan *et al.* [1], the volatility of the stock dividends is a deterministic function of time, and we introduce a Stochastic Volatility (SV) model for the rate of growth in the price of a stock. In stochastic volatility models, the volatility changes randomly over time according to some stochastic differential equations or some discrete-time random processes, which have been widely used in financial economics and mathematical finance.

Many empirical studies have shown that stochastic volatility models often provide a more realistic description than Constant Volatility (CV) models — e.g. see Taylor [13, 14]. Indeed, many theories in mathematical finance are built on continuous-time models, into which stochastic volatility models tend to fit naturally for a wide array of applications — e.g. the pricing of currencies or options and other derivatives, or in modeling the term structure of interest rates. For example, Pillay & O' Hara [9] considered the pricing of European options when the underlying asset follows a mean reverting log-normal process with stochastic volatility, and Zvan *et al.* [18] developed penalty methods for American options with stochastic volatility. Some excellent surveys on stochastic volatility models include Ghysels *et al.* [4] and Shephard [11], and the literature is already vast and rapidly growing. Other references on continuous-time stochastic volatility models include Chen [3], Hull & White [6], Kilin [7], and Wiggins [15].

In this article, we analyze the implications of the heteroscedastic effect in a continuous-time Stochastic Volatility model (SV) for evaluating risk in the cash management problem, by comparing the terminal wealth processes derived from the SV and CV models. We formulate the optimal cash management problem as a stochastic optimal control problem and solve it numerically using dynamic programming, although there are other approaches — e.g. a genetic algorithm as in Ref. [17], to solve the asset allocation problem. In Section 2, we describe the cash management problem and derive the Hamilton-Jacobi-Bellman (HJB) second-order nonlinear partial differential equation governing the value function of the optimization problem. The successive approximation algorithm introduced by Chang & Krishna [2] we employ to solve the HJB equation is discussed in Section 3, where our main results and some sensitivity analyses are also presented. Brief Concluding Remarks are made in Section 4.