

Conjugate Gradient Method for Estimation of Robin Coefficients

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Abstract. We consider a Robin inverse problem associated with the Laplace equation, which is a severely ill-posed and nonlinear. We formulate the problem as a boundary integral equation, and introduce a functional of the Robin coefficient as a regularisation term. A conjugate gradient method is proposed for solving the consequent regularised nonlinear least squares problem. Numerical examples are presented to illustrate the effectiveness of the proposed method.

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Key words: Robin inverse problem, ill-posedness, boundary integral equation, regularisation, conjugate gradient method.

1. Introduction

Let Ω be a bounded domain in R^2 with smooth boundary $\partial\Omega = \Gamma$. Consider the following Robin boundary value problem for the Laplace equation:

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + pu = g, & \text{on } \partial\Omega = \Gamma, \end{cases} \quad (1.1)$$

where ν is the unit outward normal direction on Γ , $p = p(x)$ is the Robin coefficient with support contained in $\Gamma_1 \subset \Gamma$, and $g = g(x)$ is a given input function. In recent years, many results have been developed on important properties of the forward map from p to u , such as uniqueness, continuity with respect to proper norms, and differentiability and stability in various forms [1, 6, 7, 9, 13, 14].

The related Robin inverse problem involves recovering the Robin coefficient p from a partial boundary measurement of function u — i.e. by using $u = u_0$ on a part Γ_0 of the

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boundary, where $\Gamma_0 \cap \Gamma_1 = \emptyset$. This kind of problem appears in various nondestructive evaluation methods, where an unknown material profile with support contained within an inaccessible part of the boundary is to be recovered using a partial boundary measurement made on an accessible part of the boundary. For example, in corrosion detection the Robin coefficient p represents the corrosion damage profile on an inaccessible part of the boundary, and u_0 is the electrostatic measurement on an accessible boundary [17, 21, 28]. In the study of so-called MOSFET semiconductor devices, p encodes information on the quality and location of the inaccessible metal-to-silicon contact window and u_0 is the voltage measurement on an accessible part of the boundary [3, 11, 24, 25].

To recover the Robin coefficient numerically, we need to express p as a linear combination of certain basis functions [4, 5] or discretise the problem [12, 19]. One can discretise the problem (1.1) directly by using a finite difference method or finite element method — e.g. see [19]; or alternatively, first reformulate (1.1) as a boundary integral equation and then discretising the resulting integral equation using the boundary element method or numerical quadratures [18, 23]. Since all the quantities involved in the Robin inverse problem are on the boundary Γ , solving the problem using a boundary integral equation seems natural. This approach has the advantage that the resulting discrete system is much smaller than the system obtained by discretising the original partial differential equation.

Fasino & Inglese [13, 14, 17] have investigated the case where $\Omega = [0, 1] \times [0, a]$, $\Gamma_1 = [0, 1] \times \{a\}$ and $\Gamma_0 = [0, 1] \times \{0\}$. Their main idea is to apply a “thin-plate approximation”, which is very easy to carry out. Lin & Fang [23] transformed the Robin inverse problem into a linear integral equation by introducing a new variable v and then imposing regularisation on v , and then derived linear least-square-based methods to estimate the Robin coefficient. Jin [18] considered solving the Robin inverse problem by using conjugate gradient (CG) methods. Following transformation of the inverse problem into an optimisation problem, two regularisation methods were considered together with CG methods — one was to terminate the iterative procedure at an appropriate step according to noise level in the data (without any penalty term and with the number of iterations serving as a regularisation parameter), and the other involved introducing a Tikhonov penalty term and running a CG method until convergence was reached. Jin & Zou [20] considered the estimation of piecewise constant Robin coefficients, by minimising the Modica-Mortola functional via a concave-convex procedure. Chaabane *et al.* [8] considered the estimation of piecewise constant Robin coefficients using the Kohn-Vogelius method. In this article, we pursue further numerical methods for the Robin inverse problem (1.1) in the boundary integral equation setting. In order to obtain a highly accurate approximation of the Robin coefficient p , we introduce a penalty term directly defined by p different from that in Ref. [23], where the penalty term is defined by a functional v (a function of p and u). A conjugate gradient method is then used to solve the regularised nonlinear least squares problem. Numerical results show that the approximate Robin coefficients obtained from the method proposed here are better than those produced by the quadratic programming model in Ref. [23].

In Section 2, we formulate the inverse problem as a boundary integral equation, and then transform the problem to an unconstrained nonlinear least squares problem based on

fit-to-data least squares and regularisation. In Section 3, we introduce a discrete presentation of the regularized least squares problem for planar domains. In Section 4, we propose a conjugate gradient method for minimisation of the regularized least squares problem. Numerical examples are presented in Section 5 to illustrate the effectiveness of our numerical method in recovering the Robin coefficient from noisy data. Finally, brief concluding remarks are given in Section 6.

2. Reformulation of the Robin Inverse Problem

We first introduce the boundary integral equation formulation for the boundary value problem (1.1). Assume that the domain Ω has a smooth boundary Γ , the closed subsets Γ_0 and Γ_1 are disjoint, and $p(t)$ is a nonnegative function on Γ with $\text{supp}(p) \subset \Gamma_1$. Consider the fundamental solution for the Laplace equation in R^2

$$\Phi(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x - y|} \quad \text{for } x \neq y .$$

By Green’s formula, the solution u to (1.1) in Ω can be represented in terms of its boundary value by

$$u(x) = - \int_{\Gamma} \left(\frac{\partial \Phi(x, y)}{\partial \nu_y} + p(y)\Phi(x, y) \right) u(y) ds_y + \int_{\Gamma} \Phi(x, y) g(y) ds_y, \quad x \in \Omega .$$

It is known that the boundary value of u satisfies the following boundary integral equation [2, 22, 26]:

$$\frac{1}{2}u(x) + \int_{\Gamma} \left(\frac{\partial \Phi(x, y)}{\partial \nu_y} + p(y)\Phi(x, y) \right) u(y) ds_y = \int_{\Gamma} \Phi(x, y) g(y) ds_y, \quad x \in \Gamma . \quad (2.1)$$

On defining

$$\begin{aligned} (\mathcal{D}u)(x) &= \int_{\Gamma} \frac{\partial \Phi(x, y)}{\partial \nu_y} u(y) ds_y, \\ (\mathcal{S}u)(x) &= \int_{\Gamma} \Phi(x, y) u(y) ds_y, \end{aligned} \quad \text{for } x \in \Gamma ,$$

Eq. (2.1) can be written succinctly as

$$\mathcal{A}(p)(u) := \left(\frac{1}{2}\mathcal{S} + \mathcal{D} \right) u + \mathcal{S}(pu) = \mathcal{S}g . \quad (2.2)$$

We note that the forward problem is to solve Eq. (2.1) or Eq. (2.2) to obtain u on Γ for given p , and the Robin inverse problem is to find p on Γ_1 such that the solution u of Eq. (2.1) matches the known u_0 on Γ_0 — i.e. $u|_{\Gamma_0} = u_0$.

Let us denote the restriction operator from Γ to Γ_0 by \mathcal{R}_0 — i.e. for u defined on Γ , \mathcal{R}_0u is defined on Γ_0 with $(\mathcal{R}_0u)(x) = u(x)$ for $x \in \Gamma_0$. Then for a given input g , the

measurement u_0 of u on Γ_0 can be expressed as $\mathcal{R}_0 u = u_0$. The solution of the inverse problem does not depend continuously on the data even if it does exist, as immediately seen from the severe ill-posedness of the Cauchy problem for the Laplace equation. Hence instead of working directly with equation (2.1), we turn to a variational formulation of the problem, thereby transforming it to a constrained minimisation problem — viz.

$$\begin{cases} \min_{p,u} \frac{1}{2} \|\mathcal{R}_0 u - u_0\|_{L^2(\Gamma_0)}^2 + \frac{\alpha}{2} J(p), \\ \text{s.t. } \mathcal{A}(p)(u) = f, \end{cases} \quad (2.3)$$

where $f = \mathcal{S}g$, $J(p)$ is a regularisation functional and $\alpha > 0$ is a regularisation parameter.

In this article we consider continuous Robin coefficients, so we choose the H^1 semi-norm of $p(x)$ as a regularisation functional:

$$J(p) = \int_{\Gamma_1} |p'(x)|^2 ds_x. \quad (2.4)$$

After solving $\mathcal{A}(p)(u) = f$ to get

$$u = \mathcal{A}(p)^{-1} f,$$

we can transform the constrained minimisation problem (2.3) into an unconstrained non-linear least-squares problem:

$$\min_p \frac{1}{2} \|\mathcal{R}_0 \mathcal{A}(p)^{-1} f - u_0\|_{L^2(\Gamma_0)}^2 + \frac{\alpha}{2} J(p). \quad (2.5)$$

3. Discretisation for Planar Domains

In this section, we give a discrete representation of the minimisation problems (2.3) and (2.5) for planar domains. For the sake of completeness here, we again give the parametrisation of Γ and the discretisation of relevant operators — cf. [23].

3.1. Parametrisation of Γ

Suppose the boundary Γ of a planar domain Ω has a regular parametrisation given by

$$(x_1, x_2) = (\phi(t), \psi(t)) \text{ for } 0 \leq t \leq L,$$

with $(\phi(0), \psi(0)) = (\phi(L), \psi(L))$. Assume also that $\phi(\cdot)$ and $\psi(\cdot)$ are both \mathcal{C}^2 functions. Define

$$K_d(t, s) = \frac{1}{2\pi} \frac{\psi'(s)(\phi(t) - \phi(s)) + \phi'(s)(\psi(t) - \psi(s))}{(\phi(s) - \phi(t))^2 + (\psi(s) - \psi(t))^2}$$

and

$$K_s(t, s) = -\frac{1}{2\pi} \left(\ln \sqrt{(\phi(s) - \phi(t))^2 + (\psi(s) - \psi(t))^2} \right) \sqrt{(\phi'(s))^2 + (\psi'(s))^2},$$

for $x = (\phi(t), \psi(t))$ and $y = (\phi(s), \psi(s))$ on Γ with $x \neq y$ ($t \neq s$). We can write the operators \mathcal{D} and \mathcal{S} as

$$\begin{aligned} (\mathcal{D}u)(t) &= \int_0^L K_d(t,s)u(s)ds, \\ (\mathcal{S}u)(t) &= \int_0^L K_s(t,s)u(s)ds, \end{aligned}$$

where $u(s) = u(\phi(s), \psi(s))$.

3.2. Discretisation.

We use the mid-point quadrature rule to discretise the integral operators and central difference quotients to approximate derivatives. If the interval $[0, L]$ be partitioned into n uniform subintervals $[(i-1)h, ih]$, $i = 1, 2, \dots, n$ where $h = L/n$, then the quadrature points are $t_i = (i-1/2)h$, $i = 1, 2, \dots, n$. Suppose

$$\{t_i\}_{i=1}^n \cap \{t : (\phi(t), \psi(t)) \in \Gamma_0\} = \{t_{m_1+1}, \dots, t_{m_2}\}$$

and

$$\{t_i\}_{i=1}^n \cap \{t : (\phi(t), \psi(t)) \in \Gamma_1\} = \{t_{m_3+1}, \dots, t_{m_4}\}.$$

Let \mathbf{u} and \mathbf{p} denote the discretised functions of $u(t)$ on Γ and $p(t)$ on Γ_1 , respectively — i.e.

$$\mathbf{u} = [u(t_1), \dots, u(t_n)]^T, \quad \mathbf{p} = [p(t_{m_3+1}), \dots, p(t_{m_4})]^T;$$

and for the discrete data

$$\mathbf{u}_0 = [u_0(t_{m_1+1}), \dots, u_0(t_{m_2})]^T, \quad \mathbf{f} = [f(t_1), \dots, f(t_n)]^T.$$

If the matrix representation of the operator \mathcal{D} is denoted by D and that of \mathcal{S} by S , then the matrix representation of $\mathcal{A}(p)$ in (2.2) is

$$A(\mathbf{p}) = \frac{1}{2}I + D + SP, \quad (3.1)$$

where I is the identity matrix of order n and

$$P = \text{diag} \left([O_{m_3 \times 1}; \quad \mathbf{p}; \quad O_{(n-m_4) \times 1}] \right).$$

Here $O_{m \times 1}$ denotes the zero column of length m . As for the regularisation term defined by (2.4), incorporating the factor $1/h$ into the regularisation parameter α we approximate it by

$$\begin{aligned} J(\mathbf{p}) &= (p_{m_3+1} - 0)^2 + (p_{m_3+2} - p_{m_3+1})^2 + \dots + (p_{m_4} - p_{m_4-1})^2 + (0 - p_{m_4})^2 \\ &= (p_{m_3+1})^2 + \sum_{i=m_3+2}^{m_4} (p_i - p_{i-1})^2 + (p_{m_4})^2, \end{aligned}$$

as $(\phi(t_{m_3}), \psi(t_{m_3})), (\phi(t_{m_4+1}), \psi(t_{m_4+1})) \notin \Gamma_1$, $p_{m_3} = p(t_{m_3}) = 0$, $p_{m_4+1} = p(t_{m_4+1}) = 0$ — i.e.

$$J(\mathbf{p}) = \mathbf{p}^T T \mathbf{p},$$

where

$$T = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}_{m_4-m_3}.$$

Thus the discrete representation of the nonlinear least-squares problem (2.5) is given by

$$\min_{\mathbf{p}} \frac{1}{2} \|R_0 A(\mathbf{p})^{-1} \mathbf{f} - \mathbf{u}_0\|^2 + \frac{\alpha}{2} J(\mathbf{p}), \quad (3.2)$$

where R_0 is the discretisation of the restriction operator \mathcal{R}_0 — i.e.

$$R_0 = \begin{bmatrix} O_{(m_2-m_1) \times m_2} & I_{m_2-m_1} & O_{(m_2-m_1) \times (n-m_2)} \end{bmatrix},$$

where $O_{r \times s}$ denotes the $r \times s$ zero matrix and I_r the identity matrix of order r .

3.3. Matrices D and S for elliptic domains.

Suppose Ω is an ellipse in R^2 :

$$\Omega = \left\{ (x_1, x_2) : \left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 < 1 \right\}$$

with $a, b > 0$. The usual parametrisation for Γ is

$$(x_1, x_2) = (\phi(t), \psi(t)) = (a \cos(2\pi t), b \sin(2\pi t)) \quad \text{for } 0 \leq t \leq 1.$$

In this case, the kernel in the integral operator \mathcal{D} is

$$K_d(t, s) = -\frac{ab}{2(a^2 \sin^2(\pi(t+s)) + b^2 \cos^2(\pi(t+s)))},$$

and since $K_d(t, s)$ is smooth we have

$$D = h \left[K_d(t_i, t_j) \right]_{i,j=1}^n.$$

The kernel in the integral operator \mathcal{S} is weakly singular at $s = t$, $(t, s) = (0, 1)$ and $(1, 0)$ — so should be taken into account in discretising $\int_0^1 K_s(t, s)u(s)ds$ to avoid large errors. Thus by decomposing $K_s(t, s)$ as

$$K_s(t, s) = (K_{s1}(t, s) + K_{s2}(t, s))K_{s3}(t, s),$$

where

$$\begin{aligned} K_{s1}(t, s) &= \ln (2|\sin(\pi(t-s))|) , \\ K_{s2}(t, s) &= \ln \sqrt{a^2 \sin^2(\pi(t+s)) + b^2 \cos^2(\pi(t+s))} , \\ K_{s3}(s) &= -\sqrt{a^2 \sin^2(2\pi(s)) + b^2 \cos^2(2\pi(s))} , \end{aligned}$$

on using the singularity subtraction technique (e.g. see [10]) we obtain

$$S = \left[hK_s(t_i, t_j) - \delta_{i,j} h \sum_{k=1}^n K_{s1}(t_i, t_k) K_{s3}(t_k) \right]_{i,j=1}^n ,$$

involving the Kronecker delta

$$\delta_{i,j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

4. A Conjugate Gradient Method

We solve the regularised minimisation problem (3.2) by the well-known Conjugate Gradient (CG) method, one of the most popular quasi-Newton methods for solving non-linear optimisation problems without constraints. (It has been proven that the CG method performs well even for non-smooth objective functions [27].) In order to do so, we need to compute the gradients of $F(\mathbf{p}) := \frac{1}{2} \|R_0 A(\mathbf{p})^{-1} \mathbf{f} - \mathbf{u}_0\|_2^2$ and $J(\mathbf{p})$. Since

$$\begin{aligned} \frac{\partial}{\partial p_i} R_0 A(\mathbf{p})^{-1} \mathbf{f} &= \lim_{\tau \rightarrow 0} R_0 [A(\mathbf{p} + \tau \mathbf{e}_i)^{-1} - A(\mathbf{p})^{-1}] \mathbf{f} / \tau \\ &= \lim_{\tau \rightarrow 0} R_0 A(\mathbf{p} + \tau \mathbf{e}_i)^{-1} \cdot [A(\mathbf{p}) - A(\mathbf{p} + \tau \mathbf{e}_i)] / \tau \cdot A(\mathbf{p})^{-1} \mathbf{f} \\ &= -R_0 A(\mathbf{p})^{-1} S \operatorname{diag}(\delta_{m_3+i,1}, \dots, \delta_{m_3+i,n}) A(\mathbf{p})^{-1} \mathbf{f}, \end{aligned}$$

where \mathbf{e}_i is the i th column of the identity matrix of order n , we have

$$\frac{\partial}{\partial p_i} R_0 A(\mathbf{p})^{-1} \mathbf{f} = -R_0 A(\mathbf{p})^{-1} S (0, \dots, 0, u_{m_3+i}, 0, \dots, 0)^T, \quad (4.1)$$

on identifying $A(\mathbf{p})^{-1} \mathbf{f}$ with $\mathbf{u}_p = (u_1, \dots, u_n)^T$. It follows that

$$\frac{\partial}{\partial p_i} F(\mathbf{p}) = -(0, \dots, 0, u_{m_3+i}, 0, \dots, 0) S^T A(\mathbf{p})^{-T} R_0^T (R_0 \mathbf{u}_p - \mathbf{u}_0),$$

so we get

$$\nabla F(\mathbf{p}) = -\operatorname{diag}(u_{m_3+1}, \dots, u_{m_4}) S_1^T A(\mathbf{p})^{-T} R_0^T (R_0 \mathbf{u}_p - \mathbf{u}_0),$$

where $S_1 = S(\cdot, m_3 + 1 : m_4)$. We note that the gradient of $J(\mathbf{p}) = \mathbf{p}^T T \mathbf{p}$ is given by $\nabla J(\mathbf{p}) = 2T \mathbf{p}$, and it follows that the gradient of the objective function of the minimisation (3.2) is given by

$$\mathbf{g} = -\operatorname{diag}(u_{m_3+1}, \dots, u_{m_4}) S_1^T A(\mathbf{p})^{-1} R_0^T (R_0 A(\mathbf{p})^{-1} \mathbf{f} - \mathbf{u}_0) + \alpha T \mathbf{p}. \quad (4.2)$$

Before we proceed to the Conjugate Gradient method, we discuss how to get a good approximate step length in line search. Let \mathbf{p} be the current approximate Robin coefficient and \mathbf{d} the search direction. From Eq. (4.1) we see that the Jacobian of $R_0A(\mathbf{p})^{-1}\mathbf{f}$ is $-R_0A(\mathbf{p})^{-1}S_1\text{diag}(u_{m_3+1}, \dots, u_{m_4})$. Letting

$$\delta\mathbf{u}_d = -A(\mathbf{p})^{-1}S_1\text{diag}(u_{m_3+1}, \dots, u_{m_4})\mathbf{d} \quad \text{and} \quad \mathbf{r} = R_0A(\mathbf{p})^{-1}\mathbf{f} - \mathbf{u}_0 = R_0\mathbf{u}_p - \mathbf{u}_0, \quad (4.3)$$

we have

$$\begin{aligned} R_0A(\mathbf{p} + \gamma\mathbf{d})^{-1}\mathbf{f} - \mathbf{u}_0 &\approx R_0A(\mathbf{p})^{-1}\mathbf{f} - \gamma R_0A(\mathbf{p})^{-1}S_1\text{diag}(u_{m_3+1}, \dots, u_{m_4})\mathbf{d} - \mathbf{u}_0 \\ &= \gamma R_0\delta\mathbf{u}_d + \mathbf{r}, \end{aligned}$$

whence

$$\begin{aligned} &F(\mathbf{p} + \gamma\mathbf{d}) + \frac{\alpha}{2}J(\mathbf{p} + \gamma\mathbf{d}) \\ &\approx \frac{1}{2}\|\gamma R_0\delta\mathbf{u}_d + \mathbf{r}\|^2 + \frac{\alpha}{2}(\mathbf{p} + \gamma\mathbf{d})^T T(\mathbf{p} + \gamma\mathbf{d}) \\ &= \frac{1}{2}\left((R_0\delta\mathbf{u}_d)^T(R_0\delta\mathbf{u}_d) + \alpha\mathbf{d}^T T\mathbf{d}\right)\gamma^2 + (\mathbf{r}^T(R_0\delta\mathbf{u}_d) + \alpha\mathbf{d}^T T\mathbf{p})\gamma + \frac{1}{2}\left(\mathbf{r}^T\mathbf{r} + \frac{\alpha}{2}\mathbf{p}^T T\mathbf{p}\right), \end{aligned}$$

hence a good approximate step length in the direction \mathbf{d} is

$$\tilde{\gamma} = -\frac{\mathbf{r}^T(R_0\delta\mathbf{u}_d) + \alpha\mathbf{d}^T T\mathbf{p}}{(R_0\delta\mathbf{u}_d)^T(R_0\delta\mathbf{u}_d) + \alpha\mathbf{d}^T T\mathbf{d}}. \quad (4.4)$$

We now present the CG method, which is similar to the one presented in Ref. [18].

Algorithm 1. Conjugate Gradient Method

- (a) Choose \mathbf{p}_0 , and set $k = 0$.
 - (b) Compute \mathbf{u}_p by solving the direct problem (2.2) with $\mathbf{p} = \mathbf{p}_k$, and compute the residual $\mathbf{r}_k = R_0\mathbf{u}_p - \mathbf{u}_0$.
 - (c) Compute the gradient \mathbf{g}_k by using (4.2) and calculate the descent direction $\mathbf{d}_k = -\mathbf{g}_k + \beta_{k-1}\mathbf{d}_{k-1}$ with the conjugate coefficient β_{k-1} being given by
$$\beta_{k-1} = \frac{\|\mathbf{g}_k\|_2^2}{\|\mathbf{g}_{k-1}\|_2^2} \quad (\text{assume that } \beta_{-1} = 0).$$
 - (d) Solve $A(\mathbf{p}_k)\delta\mathbf{u}_{d_k} = S_1\text{diag}(u_{m_3+1}, \dots, u_{m_4})\mathbf{d}_k$ to get $\delta\mathbf{u}_{d_k}$; see (4.3).
 - (e) Calculate an approximate step length $\tilde{\gamma}_k$ in the direction \mathbf{d}_k :
$$\tilde{\gamma}_k = -\frac{\mathbf{r}_k^T(R_0\delta\mathbf{u}_{d_k}) + \alpha\mathbf{d}_k^T T\mathbf{p}_k}{(R_0\delta\mathbf{u}_{d_k})^T(R_0\delta\mathbf{u}_{d_k}) + \alpha\mathbf{d}_k^T T\mathbf{d}_k}; \quad \text{see (4.4)}.$$
 - (f) Determine the step length γ_k by minimising the function $F(\mathbf{p}_k + \gamma\mathbf{d}_k) + \alpha J(\mathbf{p}_k + \gamma\mathbf{d}_k)$ with $\tilde{\gamma}_k$ as the initial guess.
 - (f) Update the Robin coefficient: $\mathbf{p}_{k+1} = \mathbf{p}_k + \gamma_k\mathbf{d}_k$.
 - (g) Increase k by one and go to step (b), r and then repeat the above procedure until a stopping criterion is satisfied.
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Remark 4.1. The difference between Algorithm 1 (henceforth denoted by CG1) and the one presented in [18] (denoted by CG2) is the method used for line search. Thus CG2 uses $\tilde{\gamma}_k$ as the step length, while CG1 uses $\tilde{\gamma}_k$ as the initial guess of the step length and then carries out an exact line search (we use the Matlab function $fminunc()$ to seek the step length γ_k). It is obvious that CG1 involves more computational cost in each step than CG2, and our numerical tests described below show that CG1 requires less iterations and also produces better results.

5. Numerical Examples

We now discuss the results of applying our CG1 method to solve the inverse problem corresponding to (1.1) on elliptic domains. For simplicity, in our numerical tests below we assume the ellipse has the standard parametrisation

$$x = x(t) = (a \cos(2\pi t), b \sin(2\pi t)), \quad 0 \leq t \leq 1,$$

where $a = 1$ and b is a parameter to be chosen. The two segments Γ_0 and Γ_1 are chosen as

$$\begin{aligned} \Gamma_0 &= \{(a \cos(2\pi t), b \sin(2\pi t)) : t \in [0.55, 0.85]\} \\ &\quad \text{(a centred segment on the bottom half),} \\ \Gamma_1 &= \{(a \cos(2\pi t), b \sin(2\pi t)) : t \in [0.15, 0.45]\} \\ &\quad \text{(a centred segment on the top half).} \end{aligned}$$

The function $g(t)$ is chosen as

$$g(a \cos(2\pi t), b \sin(2\pi t)) = \begin{cases} 1, & \text{if } t \in [0.4, 0.6], \\ 0, & \text{elsewhere on } \Gamma. \end{cases}$$

We use two profiles for the Robin coefficient $p(t)$ that previously have been tested in the literature — cf. [9, 23] for example.

Our tests were carried out by using Matlab. We set uniform partitions on the parameter interval $[0, 1]$ with $h = 0.0025$. In all of our tests below, we first chose $p(t)$ and obtained approximate values of the solution to (2.2) at the grid points t_1, t_2, \dots, t_n by solving a corresponding discrete system

$$A(\mathbf{p})\mathbf{u} = \mathbf{f}$$

by using Gauss elimination, where $A(\mathbf{p})$ is defined in Eq. (3.1). Then we produced the data u_0 from $u|_{\Gamma_0}$ with a certain level of random noise added — i.e.

$$u_0(t_i) = u_i + 2\delta(\text{rand} - 0.5), \quad i = m_1 + 1, \dots, m_2,$$

where the symbol ‘rand’ denotes a random number from the uniform distribution of interval $(0, 1)$, and δ is an absolute noise level. In all of our tests, we always chose the initial guess to be $\mathbf{p}_0 = (0.5, 0.5, \dots, 0.5)^T$.

It is known that regularisation parameters play an important role in solving ill-posed problems, and the discrepancy principle can be used to obtain a valid regularisation parameter when an accurate estimate of the noise level of the data is available. However, it may be that the value of the parameter α can vary in a slightly larger range — cf. Example 5.1 below. In our numerical experiments, we assumed that an accurate estimate of $E := \|R_0\mathbf{u} - \mathbf{u}_0\|$ is known. We first chose an initial α , and then updated the value of α according to the value of $\|R_0\mathbf{u}_p - \mathbf{u}_0\|_2$. More precisely, if $\|R_0\mathbf{u}_p - \mathbf{u}_0\|_2 > 2E$, we replaced α by 0.5α ; and if $\|R_0\mathbf{u}_p - \mathbf{u}_0\|_2 < E$, we replaced α by 2α . Finally, we modified the parameter α slightly to get a best possible estimate of \mathbf{p} .

Example 5.1 (Convergence process of CG1 and effect of regularisation parameter). We set $b = 0.2$ and noise level $\delta = 10^{-3}$. In this case, the relative error in the input data is about 2.6×10^{-4} (we estimated the error by comparing \mathbf{u}_0 with a numerical solution $\tilde{\mathbf{u}}$ obtained using a fine grid). At each step of the CG1 method, we computed the error of the k th approximate Robin coefficient

$$err(k) = \|\mathbf{p}_k - \mathbf{p}_{\text{true}}\|_2,$$

where \mathbf{p}_{true} denotes the true Robin coefficient, and the difference between $\mathbf{u}_{\mathbf{p}_k}$ (determined by the approximate Robin coefficient \mathbf{p}_k) and the measure data the \mathbf{u}_0 — viz.

$$E(k) = \|R_0\mathbf{u}_{\mathbf{p}_k} - \mathbf{u}_0\|_2.$$

In Fig. 1, $err(k)$ and $E(k)$ are plotted as functions of the number of iterations k . We can see that both $err(k)$ and $E(k)$ decreased steadily in the early iterations, up to a certain iteration, and then stalled. Moreover, the quality of the recovered Robin coefficient has no significant difference for the value of α over a certain range.

Assuming that $g \geq 0$ and does not vanish identically, then $u > 0$ over the boundary Γ , which consequently guaranteed the identifiability of the Robin coefficient p by the maximum principle ([16]). However, the approximate \mathbf{p} obtained by using the CG1 method may have components that are negative (see Fig. 2), which does not meet the condition $p_i \geq 0$. There are many methods for handling this problem — e.g. see [4, 15]. In our numerical tests, we simply replaced p_i by $\max\{p_i, 0\}$ for the final results from the CG1 method.

Example 5.2 (Comparison with a QP method). We compared the performance of the CG1 method with the quadratic programming (QP) method proposed in Ref. [23], which is a very accurate method for recovering the Robin coefficient.

We set the noise level to $\delta = 10^{-3}$ and considered two profiles, which are shown together with reconstructed profiles in Fig. 3. We note that the relative errors in the input data for the above profiles are 2.6×10^{-4} and 3.4×10^{-4} , respectively. We also observe that the recovered profiles obtained using the CG1 method are more accurate than the one recovered using the QP method. However, we point out that the regularisation term we introduced is a functional of the profile $p(x)$, while the one proposed in Ref. [23] is a functional of $v(x) = (\bar{p}(x) - p(x))u(x)$, where $\bar{p}(x)$ is an upper bound of $p(x)$. We also

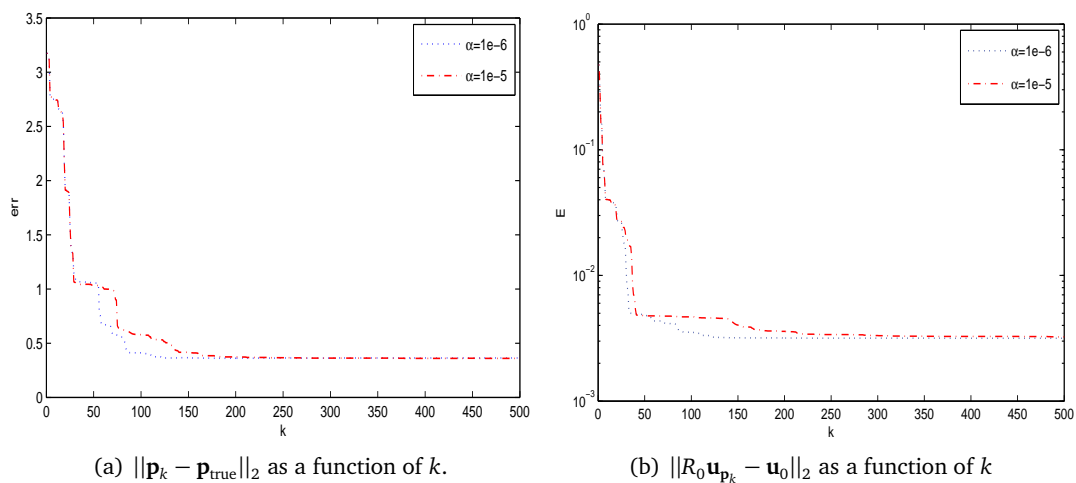


Figure 1: The convergence process.

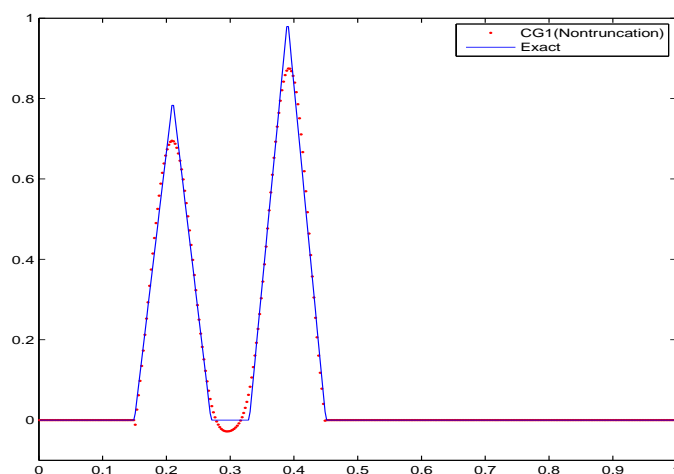
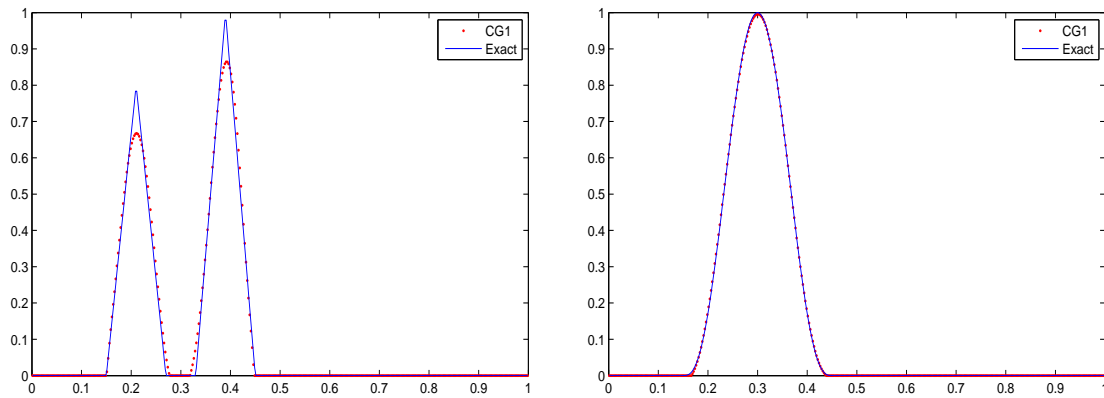


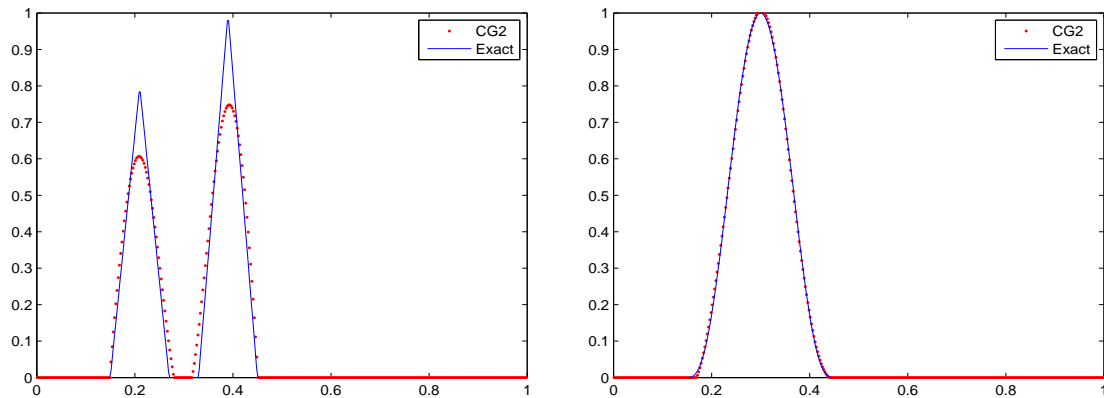
Figure 2: A recovered Robin coefficient without truncation.

note that the recovered profiles obtained using the CG1 are better than those obtained using CG2, which indicates that the recovered profiles are sensitive to minor details (the only difference between CG1 and CG2 is in the line search).

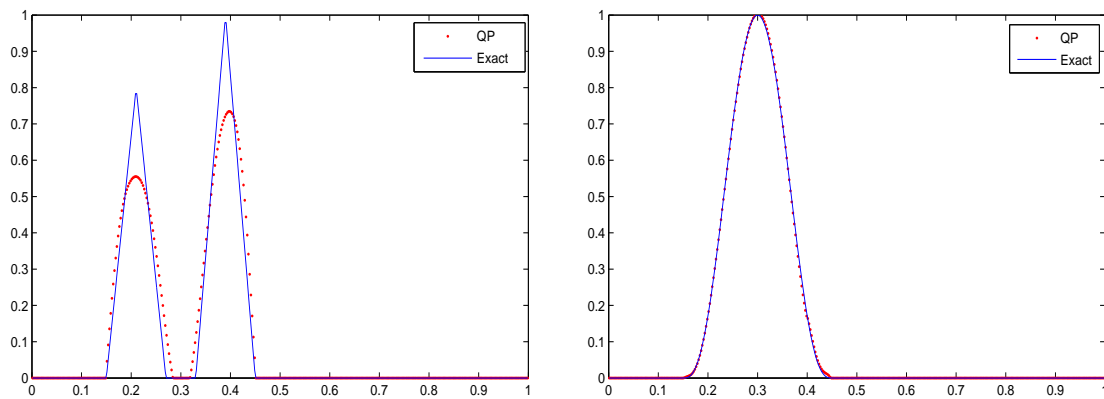
Example 5.3 (Effect of noise level). In this example, we investigated the effect of the noise level on the performance of the proposed method for reconstructing the Robin coefficient \mathbf{p} . We used the elliptic domain with $a = 1$, $b = 0.1$. The Robin coefficients recovered from noisy data with $\delta = 10^{-2}, 10^{-3}, 10^{-4}$ are plotted in Fig. 4. (The relative errors in the input data for the above two profiles are 3.0×10^{-3} and 3.3×10^{-3} , 2.9×10^{-4} and 3.2×10^{-4} , 4.1×10^{-5} and 3.7×10^{-5} , respectively.) As expected, the quality of the reconstructed



(a) Recovered profiles obtained using CG1.



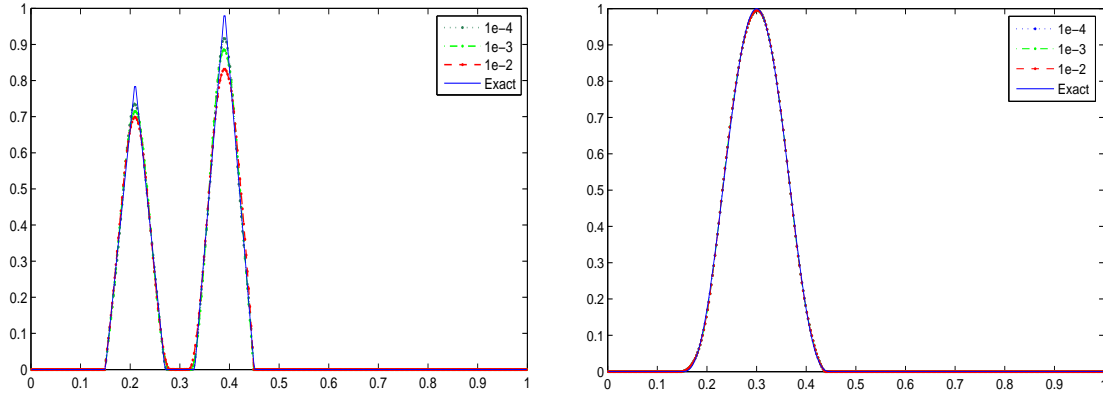
(b) Recovered profiles obtained using CG2.



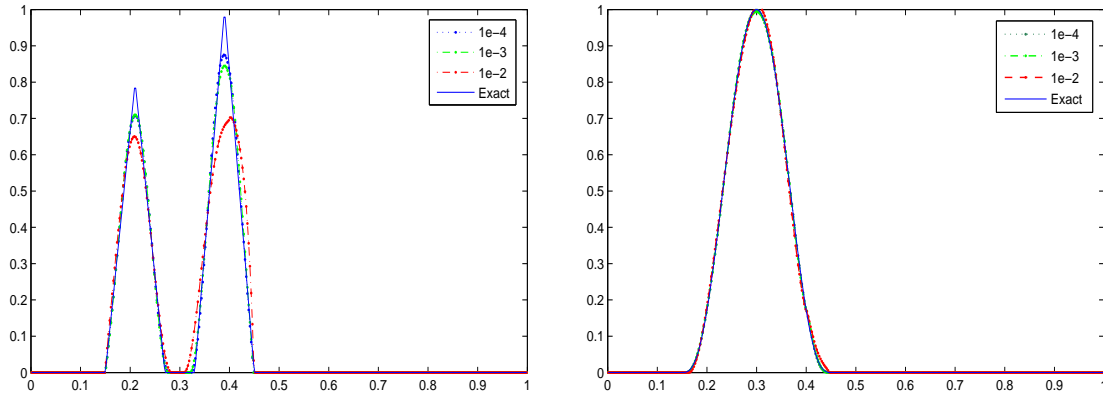
(c) Recovered profiles obtained using QP.

Figure 3: Recovered profiles obtained using CG1, CG2 and QP for $a = 1$, $b = 0.2$ and $\delta = 10^{-3}$.

Robin coefficient worsens as the noise level increases, but the CG1 method proved more tolerant to noise than the QP method.



(a) Recovered profiles obtained using CG1.



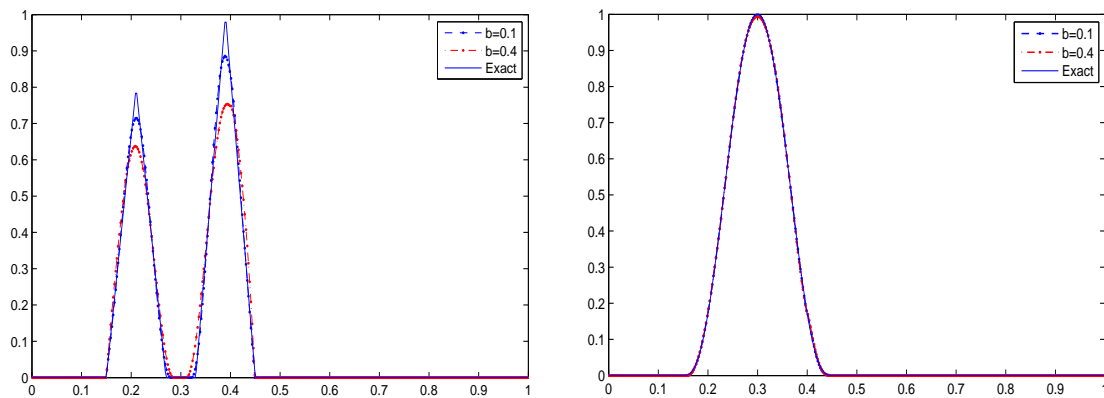
(b) Recovered profiles obtained using QP

Figure 4: Effect of the noise level δ in the data on the different methods with $b = 0.1$.

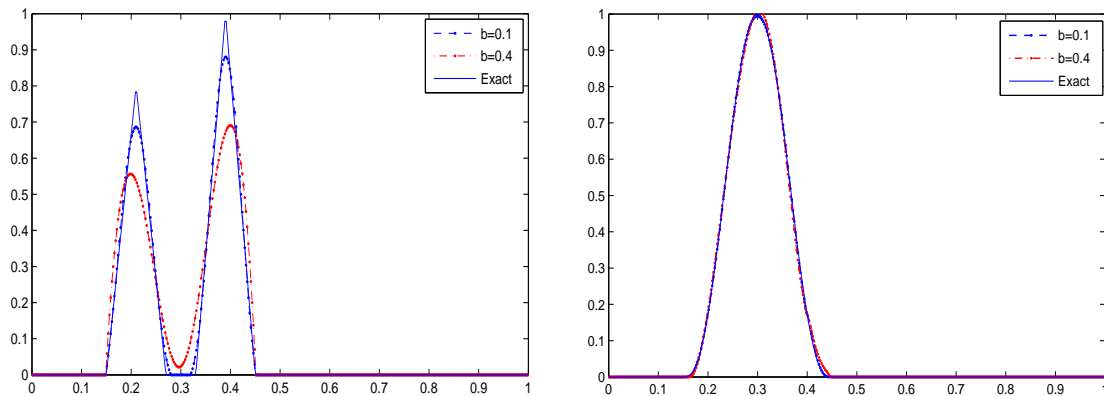
Example 5.4 (Effect of parameter b). In this example, we fixed $a = 1$ and varied b to illustrate the effect of parameter b on the recovered profiles. The noise level was set to $\delta = 10^{-3}$. The results for the two \mathbf{p} profiles are presented in Fig. 5. Although we see that the recovered \mathbf{p} becomes less accurate as b increases, the results obtained using the CG1 method are still reasonably good, as we can obtain satisfactory reconstruction of \mathbf{p} for both profiles. Moreover, we see from the left column of Fig. 5 that the CG1 method performed much better than the QP method when b was large.

6. Concluding Remarks

The Robin inverse problem considered is a severely ill-posed and nonlinear, where numerical methods have attracted a lot of attention recently. We formulated our Robin inverse problem in the setting of boundary integral equations as in Ref. [23], and then introduced a functional of the Robin coefficient as a regularisation term. A conjugate gradient method was proposed to solve the regularised nonlinear least squares problem.



(a) Recovered profiles obtained by using CG1.



(b) Recovered profiles obtained by using QR

Figure 5: Reconstruction results with various parameter b .

Our numerical experiments have shown effectiveness and robustness in recovering the Robin coefficient.

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