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Conjugate Gradient Method for Estimation of Robin Coefficients

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Abstract. We consider a Robin inverse problem associated with the Laplace equation, which is a severely ill-posed and nonlinear. We formulate the problem as a boundary integral equation, and introduce a functional of the Robin coefficient as a regularisation term. A conjugate gradient method is proposed for solving the consequent regularised nonlinear least squares problem. Numerical examples are presented to illustrate the effectiveness of the proposed method.

AMS subject classifications: 45Q05, 65N21

Key words: Robin inverse problem, ill-posedness, boundary integral equation, regularisation, conjugate gradient method.

1. Introduction

Let Ω be a bounded domain in R^2 with smooth boundary $\partial\Omega = \Gamma$. Consider the following Robin boundary value problem for the Laplace equation:

$$\begin{cases} \Delta u = 0 \,, & \text{in } \Omega \,, \\ \dfrac{\partial u}{\partial \nu} + pu = g \,, & \text{on } \partial\Omega = \Gamma \,, \end{cases} \tag{1.1}$$

where ν is the unit outward normal direction on Γ, $p = p(x)$ is the Robin coefficient with support contained in $\Gamma_1 \subset \Gamma$, and $g = g(x)$ is a given input function. In recent years, many results have been developed on important properties of the forward map from p to u, such as uniqueness, continuity with respect to proper norms, and differentiability and stability in various forms $[1,6,7,9,13,14]$.

The related Robin inverse problem involves recovering the Robin coefficient p from a partial boundary measurement of function u — i.e. by using $u = u_0$ on a part Γ_0 of the

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boundary, where $\Gamma_0 \cap \Gamma_1 = \emptyset$. This kind of problem appears in various nondestructive evaluation methods, where an unknown material profile with support contained within an inaccessible part of the boundary is to be recovered using a partial boundary measurement made on an accessible part of the boundary. For example, in corrosion detection the Robin coefficient p represents the corrosion damage profile on an inaccessible part of the boundary, and u_0 is the electrostatic measurement on an accessible boundary [17, 21, 28]. In the study of socalled MOSFET semiconductor devices, p encodes information on the quality and location of the inaccessible metal-to-silicon contact window and u_0 is the voltage measurement on an accessible part of the boundary [3, 11, 24, 25].

To recover the Robin coefficient numerically, we need to express p as a linear combination of certain basis functions [4, 5] or discretise the problem [12, 19]. One can discretise the problem (1.1) directly by using a finite difference method or finite element method — e.g. see [19]; or alternatively, first reformulate (1.1) as a boundary integral equation and then discretising the resulting integral equation using the boundary element method or numerical quadratures [18, 23]. Since all the quantities involved in the Robin inverse problem are on the boundary Γ, solving the problem using a boundary integral equation seems natural. This approach has the advantage that the resulting discrete system is much smaller than the system obtained by discretising the original partial differential equation.

Fasino & Inglese [13, 14, 17] have investigated the case where $\Omega = [0,1] \times [0,a]$, $\Gamma_1 = [0,1] \times \{a\}$ and $\Gamma_0 = [0,1] \times \{0\}$. Their main idea is to apply a "thin-plate approximation", which is very easy to carry out. Lin & Fang [23] transformed the Robin inverse problem into a linear integral equation by introducing a new variable v and then imposing regularisation on v, and then derived linear least-square-based methods to estimate the Robin coefficient. Jin [18] considered solving the Robin inverse problem by using conjugate gradient (CG) methods. Following transformation of the inverse problem into an optimisation problem, two regularisation methods were considered together with CG methods — one was to terminate the iterative procedure at an appropriate step according to noise level in the data (without any penalty term and with the number of iterations serving as a regularisation parameter), and the other involved introducing a Tikhonov penalty term and running a CG method until convergence was reached. Jin & Zou [20] considered the estimation of piecewise constant Robin coefficients, by minimising the Modica-Mortola functional via a concave-convex procedure. Chaabane *et al.* [8] considered the estimation of piecewise constant Robin coefficients using the Kohn-Vogelius method. In this article, we pursue further numerical methods for the Robin inverse problem (1.1) in the boundary integral equation setting. In order to obtain a highly accurate approximation of the Robin coefficient p, we introduce a penalty term directly defined by p different from that in Ref. [23], where the penalty term is defined by a functional v (a function of p and u). A conjugate gradient method is then used to solve the regularised nonlinear least squares problem. Numerical results show that the approximate Robin coefficients obtained from the method proposed here are better than those produced by the quadratic programming model in Ref. [23].

In Section 2, we formulate the inverse problem as a boundary integral equation, and then transform the problem to an unconstrained nonlinear least squares problem based on