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Hall MHD and Electron Inertia Effects in Current Sheet Formation at a Magnetic Neutral Line

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Abstract. An exact self-similar solution is used to investigate current sheet formation at a magnetic neutral line in incompressible Hall magnetohydrodynamics. The collapse to a current sheet is modelled as a finite-time singularity in the solution for electric current density at the neutral line. We establish that a finite-time collapse to the current sheet can occur in Hall magnetohydrodynamics, and we find a criterion for the finite-time singularity in terms of the initial conditions. We derive an asymptotic solution for the singularity formation and a formula for the singularity formation time. The analytical results are illustrated by numerical solutions, and we also investigate an alternative similarity reduction. Finally, we generalise our solution to incorporate resistive, viscous and electron inertia terms.

AMS subject classifications: 76W05, 85A30

Key words: Finite-time singularities, Hall MHD, magnetic reconnection, current sheet formation.

1. Introduction

The Hall effect can significantly modify plasma behaviour [18,42]. In particular, the magnetic reconnection rates predicted by resistive magnetohydrodynamic models [27,38] are too slow to explain reconnection in laboratory and astrophysical plasmas [1,44,46]. Numerical simulations demonstrate that including the Hall terms can speed up reconnection [2,3,12,29]. Moreover, numerical results are consistent with analytical models that quantify the role of the Hall effect in steady reconnection [24,33,41].

How quickly does a current sheet form in a weakly collisional plasma, and what is the role of the Hall effect in the process? Singularity formation models, which identify the sheet formation with the growth of the electric current density, make it possible to describe the current sheet formation using exact analytical solutions. Exact self-similar solutions in both ideal and resistive magnetohydrodynamics (MHD) have been found to exhibit both exponential growth of the current density [7,39] and finite-time collapse to a singularity [30]. The main limitation of these open-geometry solutions is that they do not

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predict the thickness of the emerging current sheet. However, the predicted exponential behaviour was confirmed by numerical simulations [16,37], and analytical arguments show that these solutions should evolve exponentially unless a singularity is driven by an imposed pressure [21].

Here we investigate a self-similar solution for current sheet formation in Hall MHD, i.e. when the Hall effect is included. The fundamental equations are presented in Section 2. In Sections 3 and 4, we generalise previous studies [23] by considering a general set of initial conditions and derive a criterion for the formation of a finite-time singularity. The new solution reduces to the exponentially evolving MHD solution upon setting the Hall term to zero. In Section 5, we discuss an alternative approach [32] to the singularity formation in Hall MHD. In Section 6, we generalise our new solution to incorporate resistive, viscous and electron inertia effects. We discuss the results in Section 7.

2. Generalised Ohm's Law and MHD Equations

The incompressible MHD equations in dimensionless form are given by a generalised Ohm's law [28]

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}=\eta\mathbf{J}+d_i(\mathbf{J}\times\mathbf{B}-\nabla p_e)+d_e^2\left[\partial_t\mathbf{J}+(\mathbf{v}\cdot\nabla)\mathbf{J}+(\mathbf{J}\cdot\nabla)\mathbf{v}\right]\,, \tag{2.1}$$

the equation of motion

$$\partial_t\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\nabla p+\mathbf{J}\times\mathbf{B}+\nu\nabla^2\mathbf{v}\,, \tag{2.2}$$

the continuity equation

$$\nabla\cdot\mathbf{v}=0\,, \tag{2.3}$$

and electromagnetic equations

$$\nabla\cdot\mathbf{B}=0\,, \tag{2.4}$$

$$\mathbf{J}=\nabla\times\mathbf{B}\,, \tag{2.5}$$

$$\nabla\times\mathbf{E}=-\partial_t\mathbf{B}\,, \tag{2.6}$$

where ∂_t denotes partial differentiation with respect to the time t, $\mathbf{v}$ is the plasma velocity, $\mathbf{B}$ the magnetic field, $\mathbf{J}$ the electric current density, $\mathbf{E}$ the electric field, and the total plasma pressure p and electron pressure p_e are scalar fields [43]. Here we use Gaussian cgs units for consistency with other theoretical studies. The length and magnetic field are scaled by typical reference values L and B_0, the velocity $\mathbf{v}$ is normalised by the Alfvén speed $v_A=B_0/\sqrt{4\pi\rho}$ where $\rho\simeq m_i n$ is the mass density (with the relation $m_e\ll m_i$ between the electron and ion masses and n their common particle number density), the time is normalised by the Alfvén time $t_A=L/v_A$, and the assumed constant resistivity η and viscosity ν by $4\pi L v_A/c^2$ and $L v_A$, respectively (where c is the speed of light). The adoption of the scalar viscosity term in the equation of motion, as opposed to a more general anisotropic viscous stress tensor, is justified in the vicinity of a magnetic null where the magnetic field