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Some Refined Eigenvalue Perturbation Bounds for Two-by-Two Block Hermitian Matrices

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Abstract. We consider eigenvalue perturbation bounds for Hermitian matrices, which are associated with problems arising in various computational science and engineering applications. New bounds are discussed that are sharper than some existing ones, including the well-known Weyl bound. Two numerical examples are investigated, to illustrate our theoretical presentation.

AMS subject classifications: 65F10

Key words: Eigenvalue perturbation, Hermitian block matrix, Weyl bound.

1. Introduction

We consider a Hermitian matrix

$$\mathscr{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^H & A_{22} \end{pmatrix}, \tag{1.1}$$

where $A_{11} \in \mathscr{C}^{p\times p}$ and $A_{22} \in \mathscr{C}^{k\times k}$. In particular, a matrix $\mathscr{A}$ with block form (1.1) is also said to be an $n \times n$ *generalised saddle point matrix*, or a *saddle point matrix* if A_{22} is a null matrix. Generalised saddle point systems have a wide variety of applications in computational science and engineering (e.g. see Ref. [3] and references therein), and the spectrum of the generalised saddle point matrix plays an important role in solving the generalised saddle system with Krylov subspace methods. We also note that some other authors have discussed the spectral analysis of saddle point matrices — e.g. see Refs. [1,7] and references therein.

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We will assume a perturbation of $\mathscr{A}$ expressed as

$$\mathscr{E} = \begin{pmatrix} E_{11} & E_{12} \\ E_{12}^H & E_{22} \end{pmatrix}, \tag{1.2}$$

where $E_{11} \in \mathscr{C}^{p\times p}$ and $E_{22} \in \mathscr{C}^{k\times k}$ are Hermitian, and write $\widetilde{\mathscr{A}} = \mathscr{A} + \mathscr{E}$. The spectrum of any matrix M is denoted by $\sigma(M)$; and throughout this article we assume that both $\mathscr{A}$ and $\widetilde{\mathscr{A}}$ are Hermitian, and that their eigenvalues are arranged in the increasing order.

A well-known perturbation bound for Hermitian matrices is as follows. Let λ_i and $\widetilde{\lambda}_i$ be the ith smallest eigenvalues of Hermitian matrices $\mathscr{A}$ and $\widetilde{\mathscr{A}}$, respectively. Then the Weyl bound

$$|\lambda_i - \widetilde{\lambda}_i| \le ||\mathscr{E}||_2 \tag{1.3}$$

is optimal without any restriction on the perturbation $\mathscr{E}$, and clearly it is independent of the structure of the matrices $\mathscr{A}$ and $\widetilde{\mathscr{A}}$. When a block diagonal matrix undergoes an off-diagonal perturbation (i.e. A_{12}, E_{11} and E_{22} are all zero), Mathias [10] provided another bound

$$|\lambda_i - \widetilde{\lambda}_i| \le \frac{||E_{12}||_2^2}{\eta_i}, \tag{1.4}$$

where

$$\eta_i = \min_{\lambda\in\sigma(A_{22})} |\lambda_i - \lambda|, \tag{1.5}$$

which was subsequently improved by Li & Li [8]. Recently, Nakatsukasa [11] presented an eigenvalue bound for a structured perturbation of the Hermitian 2×2 block matrix (1.1). Thus if λ_i and $\widetilde{\lambda}_i$ are the respective ith eigenvalues of $\mathscr{A}$ and $\widetilde{\mathscr{A}}$, then one has

$$|\lambda_i - \widetilde{\lambda}_i| \le ||E_{11}||_2 + 2||E_{12}||_2\tau_i + ||E_{22}||_2\tau_i^2 \tag{1.6}$$

provided

$$\tau_i = \frac{||E||_2 + ||E_{12}||_2}{\min_{\lambda\in\sigma(A_{22})}|\lambda_i - \lambda| - 2||\mathscr{E}||_2} > 0\,.$$

It is evident that this eigenvalue perturbation bound depends on the block structure of $\mathscr{A}$. Remark 1 in Ref. [11] suggests that $\tau_i < 1$ is necessary for a bound that is sharper than the Weyl bound (1.3), and Li *et al.* [9] improved the bound (1.6) as follows:

$$\begin{aligned}|\lambda_i - \widetilde{\lambda}_i| \le& ||E_{11}||_2 + \min\left\{1, 2\widetilde{\omega}(\delta_i^{(2)})\right\}||E_{12}||_2 \\ &+ \max\left\{||E_{22}||_2 - ||E_{11}||_2, 0\right\}\omega(\delta_i^{(2)})\widetilde{\omega}(\delta_i^{(2)})\,,\end{aligned} \tag{1.7}$$

or

$$\begin{aligned}|\lambda_i - \widetilde{\lambda}_i| \le& ||E_{22}||_2 + \min\left\{1, 2\widetilde{\omega}(\delta_i^{(1)})\right\}||E_{12}||_2 \\ &+ \max\left\{||E_{11}||_2 - ||E_{22}||_2, 0\right\}\omega(\delta_i^{(1)})\widetilde{\omega}(\delta_i^{(1)})\,,\end{aligned} \tag{1.8}$$