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# An Inexact Shift-and-Invert Arnoldi Algorithm for Large Non-Hermitian Generalised Toeplitz Eigenproblems

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**Abstract.** The shift-and-invert Arnoldi method is a most effective approach to compute a few eigenpairs of a large non-Hermitian Toeplitz matrix pencil, where the Gohberg-Semencul formula can be used to obtain the Toeplitz inverse. However, two large non-Hermitian Toeplitz systems must be solved in the first step of this method, and the cost becomes prohibitive if the desired accuracy for this step is high — especially for some ill-conditioned problems. To overcome this difficulty, we establish a relationship between the errors in solving these systems and the residual of the Toeplitz eigenproblem. We consequently present a practical stopping criterion for their numerical solution, and propose an inexact shift-and-invert Arnoldi algorithm for the generalised Toeplitz eigenproblem. Numerical experiments illustrate our theoretical results and demonstrate the efficiency of the new algorithm.

**AMS subject classifications**: 65F15, 65F10

**Key words**: Toeplitz matrix, generalised eigenproblem, shift-and-invert Arnoldi method, Gohberg-Semencul formula.

## 1. Introduction

Toeplitz matrices and operators arise in several areas of mathematics and its applications, including complex and harmonic analysis, probability theory and statistics, signal and image processing, information theory and numerical analysis [4, 5, 14]. The design of fast algorithms for Toeplitz matrices is a wide and active research field in structured numerical linear algebra, bearing in mind that a Toeplitz matrix is constant along its diagonals so only the first column and the first row of a Toeplitz matrix need be stored.

The Hermitian Toeplitz eigenproblem and the generalised Hermitian Toeplitz eigenproblem in particular have received considerable attention over recent decades — cf. [1, 13, 16, 21, 22, 25] and the references therein. Iterative methods such as the preconditioned

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Lanczos method and the shift-and-invert Lanczos method [19] are widely used to solve for the smallest eigenpair or a few extreme eigenpairs of a large Hermitian Toeplitz matrix (pencil). For instance, the sine transform-based preconditioner has been used to speed up the convergence rate of the preconditioned Lanczos method in computing the minimum eigenvalue of a symmetric positive definite Toeplitz matrix [13]. This involves only Toeplitz and sine transform matrix-vector multiplications, and hence can be computed efficiently by fast transform algorithms. Wang & Lu [22] generalised this idea and employed the sine transform-based preconditioner to compute the minimum eigenvalue of a symmetric Toeplitz matrix pencil. Recently, parallel shift-and-invert Lanczos algorithms were investigated for computing a few eigenpairs of a symmetric Toeplitz matrix, based on an efficient implementation of the discrete sine transform [1,21] and also the Gohberg-Semencul formula [6] for solving the Toeplitz linear systems involved.

Recently, special attention has been paid to the eigenproblems of large scale Toeplitz matrices [1,3,8,10,13,21,22,25]. There is an extensive literature concerning the Hermitian, the generalised Hermitian, and the non-Hermitian Toeplitz eigenproblem. However, to the best of our knowledge little attention has been paid to large non-Hermitian generalised Toeplitz eigenproblems of interest in signal processing and control theory, and on Toeplitz matrices generated by rational functions [2,16,25]. In this article, we focus on computing a few eigenpairs of the following non-Hermitian Toeplitz generalised eigenproblem:

$$A\mathbf{x} = \lambda B\mathbf{x}\,, \tag{1.1}$$

where $A$ and $B$ are large Toeplitz matrices, and the matrix pencil $(A,B)$ is regular [7,19]. When $B = I$ (the identity matrix), this generalised eigenproblem reduces to the standard Toeplitz eigenproblem.

In particular, we focus on computing a few eigenpairs of a large non-Hermitian Toeplitz matrix pencil, including extreme eigenpairs or those closest to a given shift. As previously mentioned, the shift-and-invert Arnoldi method where the Gohberg-Semencul formula [6] can be used is one of the most effective approaches for this type of problem — but two large non-Hermitian Toeplitz systems must be solved in advance, and if the desired accuracy for solving the Toeplitz linear systems is too high the computational cost becomes prohibitive. In Section 2, we establish a relationship between the errors of the Toeplitz systems and the residual of the Toeplitz eigenproblem. We then give a practical stopping criterion for solving the Toeplitz systems approximately, and propose an inexact shift-and-invert Arnoldi algorithm for the generalised Toeplitz eigenproblem. Numerical experiments discussed in Section 3 illustrate our theoretical results and demonstrate the efficiency of the new algorithm. Some concluding remarks are made in Section 4.

## 2. An Inexact Shift-and-Invert Arnoldi Method for Generalised Toeplitz Eigenproblems

The shift-and-invert technique [19] has proven a most effective method for large-scale eigenproblems. In this technique, under a shift $\sigma \in \mathbb{C}$ we have from (1.1) that

$$(A - \sigma B)\mathbf{x} = (\lambda - \sigma)B\mathbf{x}\,,$$