East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.060914.260415a

Vol. **5**, No. 2, pp. 176-191
May 2015

A Local Fractional Taylor Expansion and Its Computation for Insufficiently Smooth Functions

Zhifang Liu[1], Tongke Wang[1,*] and Guanghua Gao[2]

[1] *School of Mathematical Sciences, Tianjin Normal University, Tianjin 300387, China.*
[2] *College of Science, Nanjing University of Posts and Telecommunications, Nanjing 210023, China.*

Received 6 September 2014; Accepted (in revised version) 26 April 2015.

Abstract. A general fractional Taylor formula and its computation for insufficiently smooth functions are discussed. The Aitken delta square method and epsilon algorithm are implemented to compute the critical orders of the local fractional derivatives, from which more critical orders are recovered by analysing the regular pattern of the fractional Taylor formula. The Richardson extrapolation method is used to calculate the local fractional derivatives with critical orders. Numerical examples are provided to verify the theoretical analysis and the effectiveness of our approach.

AMS subject classifications: 26A33, 65B05

Key words: Local fractional derivative, critical order, local fractional Taylor expansion, Aitken delta square method, epsilon algorithm.

1. Introduction

In fractional calculus, different definitions of fractional derivatives lead to various forms for the corresponding fractional Taylor formulas [1,2,7,8,16,18,19,22]. There are several well-known cases such as the Riemann-Liouville fractional derivative, Grünwald fractional derivative, Caputo derivative and Weyl derivative [5,15,17,20] that are applied in different fields. Kolwankar & Gangal [11–13] introduced the notion of a local fractional derivative (LFD) when they considered continuous and nowhere differentiable Weierstrass functions, in order to avoid the difficulties that all of the above fractional derivatives are non-local and inconvenient for characterising the local fractional differential property of such functions. They also introduced a simple generalisation of the Taylor expansion. Some other researchers considered the LFD from various points of view [1–4, 14, 24]. In particular, Adda & Cresson [1,2] gave an equivalent LFD definition and also derived a special form of Taylor formula. The applications of LFD include a local fractional diffusion equation, and a

*Corresponding author. *Email addresses:* liuzhifang0628@gmail.com (Z. Liu), wangtke@sina.com (T. Wang), gaoguanghua1107@163.com (G. Gao)

 ©2015 Global-Science Press

generalised relation between stress and strain for fractal media [9]. In Ref. [23], LFD was used to derive the error asymptotic expansion for the trapezoidal rule applied to integrals with algebraic endpoint singularities, and a general fractional Taylor expansion for insufficiently smooth functions involving higher order local fractional derivatives was presented. A similar result was obtained in Ref. [10], almost simultaneously but in another way. There are also some numerical methods for fractional calculus — cf. Diethelm *et al.* [6].

In this article, we explicitly depict the Taylor expansion and proceed to compute the critical orders and the corresponding LFD. We find that a useful tool is extrapolation, which can gradually eliminate lower order terms in the asymptotic expansion of the remainder. Usually, Richardson extrapolation is preferred when the power exponents of the terms in an asymptotic expansion are known, but otherwise the Aitken Δ^2-process and ϵ—algorithm [21] based on Shanks transformation can be employed (the ϵ—algorithm extends the Aitken Δ^2-process). In Section 2, the definitions of high order LFD and the critical orders are given, and a general local Taylor expansion is discussed for different cases. The Aitken Δ^2-process and ϵ—algorithm are employed in Section 3 to calculate the first critical orders of the LFD for the local fractional Taylor expansion, and more critical orders are then recovered by analysing the regular pattern of the computed values. We also discuss the critical orders of a function derived by arithmetic or composite operations. In Section 4, the Richardson extrapolation method is implemented to obtain the local fractional derivatives effectively. Our concluding remarks are made in Section 5.

2. Local Fractional Derivative and Local Fractional Taylor Expansion

The generic term "fractional derivative" may be used to encompass both the usual integer-order and non-integer order derivatives. In this section, we briefly review the Riemann-Liouville fractional derivative and the local fractional derivative, and then define the concept of a high-order local fractional derivative associated with a general fractional Taylor formula.

Definition 2.1. (Riemann-Liouville fractional derivative) For $\alpha > 0$ and $x > a$, the α-order derivative of a function $f(x)$ is defined by

$$\frac{d^\alpha f(x)}{d(x-a)^\alpha} = \frac{1}{\Gamma(k-\alpha)}\frac{d^k}{dx^k}\int_a^x \frac{f(t)}{(x-t)^{\alpha-k+1}}dt\ ,\ k-1 \le \alpha < k\ ,\ k \in \mathbb{N}\,. \tag{2.1}$$

The Riemann-Liouville fractional derivative is defined globally on the interval $[a, x]$, so it is not accurately characterised at some points. Furthermore, the fractional derivative of a constant with arbitrary order is not equal to zero. To avoid these deficiencies, Kolwankar & Gangal [11, 13] introduced the LFD as follows.

Definition 2.2. (The KG LFD) Let $f(x) \in C^k[a, b]$. For $x_0 \in [a, b]$, if $\alpha \in (k, k+1)$ exists such that the limit

$$D^\alpha_\pm f(x_0) = \lim_{x\to x_0\pm} \frac{d^\alpha(f(x) - \sum_{i=0}^k f^{(i)}(x_0)/\Gamma(i+1)(x-x_0)^i)}{d(\pm(x-x_0))^\alpha} \tag{2.2}$$