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Multilevel Circulant Preconditioner for High-Dimensional Fractional Diffusion Equations

Siu-Long Lei, Xu Chen* and Xinhe Zhang

Department of Mathematics, University of Macau, Macau, China.

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Abstract. High-dimensional two-sided space fractional diffusion equations with variable diffusion coefficients are discussed. The problems can be solved by an implicit finite difference scheme that is proven to be uniquely solvable, unconditionally stable and first-order convergent in the infinity norm. A nonsingular multilevel circulant preconditoner is proposed to accelerate the convergence rate of the Krylov subspace linear system solver efficiently. The preconditoned matrix for fast convergence is a sum of the identity matrix, a matrix with small norm, and a matrix with low rank under certain conditions. Moreover, the preconditioner is practical, with an $O(N\log N)$ operation cost and $O(N)$ memory requirement. Illustrative numerical examples are also presented.

AMS subject classifications: 65F10, 65L12, 65T50, 26A33

Key words: High-dimensional two-sided fractional diffusion equation, implicit finite difference method, unconditionally stable, multilevel circulant preconditioner, GMRES method.

1. Introduction

Fractional diffusion equations (FDE) arise in many applications — e.g. modelling the chaotic dynamics of classical conservative systems [59], groundwater contaminant transport [2,3], turbulent flow [5,46], anomalous electrodiffusion in nerve cells [21], the foraging behaviour of animals [39] and the singling of biological cells [40], the design of photocopiers and laser printers [44], fluid and continuum mechanics [28], electrochemistry [35], biology [27], finance [38], image processing [1], and in physics [32,47].

As there are very few cases of FDE where closed-form analytical solutions are available, numerical solutions for FDE have been developed intensively, including finite difference methods [20,23,29–31,33,49–51], finite element methods [15,17,41] and spectral methods [26,45]. Nevertheless, since the fractional differential operator is nonlocal, most numerical methods for FDE generate dense coefficient matrices, and even full matrices in one-dimensional (1D) cases [56,57], which usually require a computational cost of $O(N^3)$ and storage of $O(N^2)$ where N is the size of the matrix.

*Corresponding author. *Email addresses:* `sllei@umac.mo` (S.-L. Lei), `09xchen@gmail.com` (X. Chen), `yb57010@connect.umac.mo` (X. Zhang)

Meerschaet *et al.* [29–31] proposed an implicit finite difference scheme for solving two-sided FDE, in which the shifted Grünwald-Letnikov difference approximation is employed. The difference scheme was proven to be convergent on analysing the spectral radius of the coefficient matrix of the discretised linear system, and here we show that the scheme is uniquely solvable, unconditionally stable and first-order convergent in the infinity norm. A similar result for two-dimensional (2D) FDE has been obtained in the Riesz sense [11,24], and for two-sided FDE in the 1D case [13].

To solve the nonsymmetric linear system at each time level of the difference scheme in the 1D case, Wang *et al.* found the Toeplitz-like structure of the discrete linear system of FDE, and developed a fast method with an $O(N)$ storage requirement and $O(N\log N)$ computational cost [56]. Several efficient methods using the fast Fourier transform (FFT) were then based on this Toeplitz-like structure [22,25,36]. In higher dimensions, an alternating-direction scheme for FDE with variable diffusion coefficients was proposed [55], and its unconditional stability and convergence was proven under certain conditions. The computational cost for solving the alternating-direction scheme is $O(N\log N)$ at each iteration of the Krylov subspace method, with the memory requirement $O(N)$. On the other hand, Wang *et al.* [54, 57] proposed a Krylov subspace method to solve the discretised linear system, where (due to the block Toeplitz-like structure of the coefficient matrix) each iteration of the Krylov subspace method can be done in $O(N\log N)$ operations via the FFT and the memory requirement is $O(N)$. However, the convergence rate of this Krylov subspace method is slow when the coefficient matrix is ill-conditioned, such as when the diffusion coefficients are large. In order to accelerate its convergence rate, a preconditioner is proposed in the second part of this article.

Circulant preconditioners have previously been shown to be effective for solving Toeplitz systems [7,8,34], and a Strang-based circulant preconditioner was developed successfully for solving 1D FDE problems [22]. For high-dimensional problems, on noting the tensor structure of the coefficient matrix a multilevel circulant preconditioner is proposed that is shown to be nonsingular. Since the multilevel circulant preconditioner can be diagonalised by a high-dimensional Fourier matrix, the product of the inverse of the preconditioner with any vector can be done in $O(N\log N)$ operations via a high-dimensional FFT. The operation cost in each iteration of the preconditioned Krylov subspace method is therefore $O(N\log N)$ with an $O(N)$ memory requirement. On the other hand, under certain conditions the preconditioned matrix is shown to be equal to a sum of the identity matrix, a matrix with small norm and a matrix with low rank. Fast convergence of the preconditioned Krylov subspace method can therefore be expected, and we undertook numerical tests to reveal the effectiveness of the multilevel circulant preconditioner. Furthermore, due to the similar structure of the coefficient matrix, the proposed multilevel circulant preconditioner can be applied to implicit finite difference schemes with high-order fractional derivative approximations [12,48,52,53,58,60], and to FDE with a nonlinear reaction term [24].

In Section 2, an implicit finite difference discretisation for 2D two-sided FDE is analysed, where unique solvability, unconditional stability and first-order convergence in the infinity norm are shown theoretically for any nonnegative variable diffusion coefficients. A level-2 circulant preconditioner for 2D FDE is constructed in Section 3, together with the