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Lubich Second-Order Methods for Distributed-Order Time-Fractional Differential Equations with Smooth Solutions

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Abstract. This article is devoted to the study of some high-order difference schemes for the distributed-order time-fractional equations in both one and two space dimensions. Based on the composite Simpson formula and Lubich second-order operator, a difference scheme is constructed with $\mathcal{O}(\tau^2+h^4+\sigma^4)$ convergence in the $L_1(L_\infty)$-norm for the one-dimensional case, where τ, h and σ are the respective step sizes in time, space and distributed-order. Unconditional stability and convergence are proven. An ADI difference scheme is also derived for the two-dimensional case, and proven to be unconditionally stable and $\mathcal{O}(\tau^2|\ln\tau|+h_1^4+h_2^4+\sigma^4)$ convergent in the $L_1(L_\infty)$-norm, where h_1 and h_2 are the spatial step sizes. Some numerical examples are also given to demonstrate our theoretical results.

AMS subject classifications: 65M10

Key words: Distributed-order time-fractional equations, Lubich operator, compact difference scheme, ADI scheme, convergence, stability.

1. Introduction

For several decades, considerable attention has increasingly been given to fractional differential equations, mainly because such equations with fractional operators can be more accurate than classical differential equations in describing physical and chemical processes with nonlocal connectivity [1–4] — e.g. a time-fractional differential equation to model anomalous diffusion in porous media. For the single-term time-fractional equation, quite extensive research works have investigated its analytic solution [5–8] and numerical methods for both one-dimensional [9–17] and two-dimensional [18–21] cases. Recently, some attention has also been given to a more general class of time-fractional distributed-order

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equations, which can be described in the Caputo sense as follows:

$$\mathscr{D}_t^w u(\mathbf{x},t) = \Delta u(\mathbf{x},t) + F(\mathbf{x},t)\,, \quad x \in \Omega\,,\ 0 < t \le T\,, \tag{1.1}$$

$$u(\mathbf{x},0) = 0\,, \quad \mathbf{x} \in \overline{\Omega}\,, \tag{1.2}$$

$$u(\mathbf{x},t)|_{\mathbf{x}\in\partial\Omega} = \psi(\mathbf{x},t)\,, \quad 0 < t \le T\,, \tag{1.3}$$

where

$$\mathscr{D}_t^w u(x,t) = \int_0^1 w(\alpha)\, {}_0^C D_t^\alpha u(x,t) d\alpha\,, \quad w(\alpha) \geqslant 0\,, \quad \int_0^1 w(\alpha) d\alpha = c_0 > 0\,,$$

$${}_0^C D_t^\alpha u(x,t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t (t-\xi)^{-\alpha} \frac{\partial u}{\partial \xi}(x,\xi) d\xi\,, & 0 \le \alpha < 1\,, \\ u_t(x,t) = 0\,, & \alpha = 1\,, \end{cases}$$

and both $F(\mathbf{x},t)$ and $\psi(\mathbf{x},t)$ are given smooth functions with $\psi(\mathbf{x},0) = 0$ where $\mathbf{x} \in \partial\Omega$. Incidentally, if an initial condition is inhomogeneous it is easy to make its right-hand side zero by a suitable transformation [22].

Some numerical methods have also been applied to the distributed-order differential equation, and a matrix solution approach has been proposed by Podlubny *et al.* [23]. Diethelm & Ford [24] presented numerical methods for solving distributed-order ordinary differential equations, where the distributed-order integral is first approximated by a quadrature formula and the resulting multi-term fractional differential equations can then be reduced to a system of single-term equations. Similar techniques for the multi-term fractional differential equation were discussed by Liu *et al.* [25]. Recently, Ye *et al.* [26] constructed an implicit difference scheme for time distributed-order Riesz space fractional diffusion on bounded domains, and proved unconditional stability and convergence by mathematical induction. Ford *et al.* [27] discussed a numerical method for a distributed-order time-fractional diffusion equation, and Morgado & Rebelo [28] presented an implicit scheme for the numerical approximation of a distributed-order time-fractional reaction-diffusion equation with a nonlinear source term. In these last three contributions, the midpoint rule was invoked to approximate the distribution integral, and the same discrete L1 formula was used to approximate the Caputo fractional derivatives involved, so the numerical accuracy of each of these schemes in the time variable was approximately first order. Katsikadelis [29] presented a numerical method for distributed-order differential equations using the trapezoidal rule, and the approximate multi-term fractional differential equation was solved similarly. However, stability and convergence were only demonstrated by numerical experiments, without proof.

It is well known that in long time simulations the global storage and computational time are typically very large for such difference schemes applied to fractional differential equations, so the construction of a high-order difference scheme is desirable to reduce computational complexity. In this article, for the one-dimensional and two-dimensional distributed-order equations with smooth solutions we propose high-order backward difference schemes, via the composite Simpson formula and a Lubich high-order operator [14,