East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.240715.050216a

Vol. **6**, No. 2, pp. 152-170
May 2016

# An Adaptive Time Stepping Method for Transient Dynamic Response Analysis

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*Received 24 July 2015; Accepted (in revised version) 5 February 2016.*

**Abstract.** An efficient adaptive time stepping method is proposed for transient dynamic response analysis, which is frequently encountered in civil engineering and elsewhere. The reduced problem following the spatial discretisation can be discretised in the time by a $C^0$-continuous discontinuous Galerkin method, and the adaptive time stepping strategy is based on optimal *a posteriori* error estimates developed using the energy method. Some numerical experiments demonstrate the effectiveness of our approach.

**AMS subject classifications**: 65M60, 65M12

**Key words**: Time stepping method, transient dynamic response, *a posteriori* error analysis, adaptive algorithm.

## 1. Introduction

It is often important to investigate the time-dependent changes of dynamical systems (their transient response), such as in structure analysis in civil engineering [7]. The underlying mathematical models typically involve a system of second order partial differential equations, which after spatial discretisation by finite element, finite difference or spectral methods reduce to a system of second order ordinary differential equations (ODE) — e.g. see Refs. [3, 12, 14, 19, 20, 27, 30].

The semi-discretised transient dynamic response problem we consider is as follows. For any real $T > 0$, find $\mathbf{u} : [0, T] \to \mathbb{R}^d$ where $d$ is the spatial dimension such that

$$\begin{cases} \mathbf{M}\mathbf{u}'' + \mathbf{C}\mathbf{u}' + \mathbf{K}\mathbf{u} = \mathbf{f}\,, & 0 < t < T\,, \\ \mathbf{u}(0) = \mathbf{u}_0\,, \\ \mathbf{u}'(0) = \mathbf{v}_0\,. \end{cases} \tag{1.1}$$

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Here the prime denotes differentiation with respect to time; $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the $d\times d$ global mass, damping and stiffness matrices of the dynamic system, respectively; $\mathbf{u}$, $\mathbf{u}'$ and $\mathbf{u}''$ are $d\times 1$ nodal displacement, velocity and acceleration vectors; and $\mathbf{f}$ is a load function from $[0,T]$ into $\mathbb{R}^d$.

There are many numerical methods for solving the ODE system (1.1). The most widely used include modal superposition [9, 17] and direct-time integration methods such as the Runge-Kutta, central difference, Houbolt, Newmark-$\beta$ and Wilson-$\theta$ procedures — e.g. see Ref. [14] and references therein. Space-time finite element methods provide another widely explored approach for solving second order time-dependent problems [11, 15, 16]. The time-discontinuous Galerkin (TDG) method [11, 18] produces solutions for the displacement and velocity fields together — but it has the disadvantage that an ill-conditioned $4\times 4$ block system must be solved at each time step, which is time consuming. To overcome this drawback, some $C^0$-continuous time stepping methods were introduced, where the primal variables are involved but only a $2\times 2$ block system has to be solved at each time step [22, 23]. Based on this approach, an adaptive method was proposed for solving second order abstract evolution problems where optimal *a posteriori* error estimates are established [13]. In conjunction with an error equi-distribution strategy and some ideas from the Runge-Kutta-Felberg method, an adaptive time stepping method was produced. In passing, we note that some efficient adaptive time-stepping methods were introduced to simulate the long time dynamics in the molecular beam epitaxy (MBE) model and for the Cahn-Hilliard equations [10, 29, 32, 33].

Here we extend the work in Ref. [13] to develop an adaptive time stepping method for the problem (1.1). Our new contribution is twofold. Although the time discretisation is the same, the derivation of optimal *a posteriori* error estimates is greatly simplified by using the Hermite interpolation polynomial instead of the Legendre polynomial. Our adaptive time stepping method developed from these error estimates is then shown to solve some practical problems very well. The time stepping finite element method for transient dynamic response analysis is discussed in Section 2, and in Section 3 the *a posteriori* error analysis is given in detail. The adaptive algorithm based on the *a posteriori* error estimators is presented in Section 4, followed by its application to the practical problems in Section 5 and our concluding remarks in Section 6.

## 2. The Time Stepping Finite Element Method

Throughout we assume that $\mathbf{M}$ and $\mathbf{K}$ in problem (1.1) are symmetric positive definite matrices, and that $\mathbf{C}$ is a positive semi-definite matrix. We suppose that

$$\mathbf{u}_0,\ \mathbf{v}_0 \in \mathbb{R}^d, \qquad \mathbf{f}\in L^2(0,T;\mathbb{R}^d). \tag{2.1}$$

We use a standard time stepping method to discretise problem (1.1) — e.g. see [13,22,23]. Thus we first partition the time interval $I := (0,T)$ with the nodes

$$0 = t_0 < t_1 < \cdots < t_N = T$$