East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.090615.060216a

Vol. **6**, No. 2, pp. 171-191
May 2016

Stochastic Collocation via l_1-Minimisation on Low Discrepancy Point Sets with Application to Uncertainty Quantification

Yongle Liu[1] and Ling Guo[2,*]

[1] *Department of Mathematics, Shanghai Normal University, Shanghai, China.*
[2] *Department of Mathematics and E-Institute of Shanghai Universities and Scientific Computing, Shanghai Normal University, Shanghai, China.*

Received 9 June 2015; Accepted (in revised version) 6 February 2016.

Abstract. Various numerical methods have been developed in order to solve complex systems with uncertainties, and the stochastic collocation method using ℓ_1-minimisation on low discrepancy point sets is investigated here. Halton and Sobol' sequences are considered, and low discrepancy point sets and random points are compared. The tests discussed involve a given target function in polynomial form, high-dimensional functions and a random ODE model. Our numerical results show that the low discrepancy point sets perform as well or better than random sampling for stochastic collocation via ℓ_1-minimisation.

AMS subject classifications: 41A10, 60H35, 65N35

Key words: Stochastic collocation, Quasi-Monte Carlo sequence, low discrepancy point sets, Legendre polynomials, ℓ_1-minimisation.

---

1. Introduction

In recent years, the field of Uncertainty Quantification (UQ) has received intense attention, and various numerical methods have been developed for complex systems with uncertainties. One approach is to consider the input data as random variables, or random fields with known covariance structures, so the underlying challenge is the approximation of target quantities of interest that depend on a large number of random variables. A traditional approach is the Monte Carlo method, where one first generates a number of random realisations for the prescribed random inputs and then utilises existing deterministic solvers for each realisation — e.g. see Ref. [11] and the references therein. However, the convergence of the Monte Carlo method is relatively slow.

*Corresponding author. *Email address:* `lguo@shnu.edu.cn` (L. Guo)

Stochastic collocation based on generalised polynomial chaos (gPC) is another popular approach in the computation of UQ. It is an easily implemented non-intrusive method, and leads naturally to the solution of uncoupled deterministic problems — cf. Refs. [21, 22, 26, 27] and references therein for more detail. The key step is to determine the coefficients of the gPC expansion for the solution, for which there are two major approaches — one based on interpolation and the other on regression. It is a challenge to deal with high dimensional random spaces where the number of collocation points grows fast, known as *the curse of dimensionality*.

Compressive sensing (also known as compressed sampling) in signal processing has recently been introduced to stochastic collocation methods for UQ and some key properties are obtained, such as the probability under which the sparse random response function can be recovered [9, 19]. Compressive sensing (CS) deals with insufficient information about the target function, as is the case in many practical UQ problems. For instance, solutions to linear elliptic partial differential equations with high-dimensional random coefficients admit sparse representations with respect to the gPC basis under some mild conditions [5, 28]. Using CS, one can employ arbitrary nodal sets with an arbitrary number of nodes, which can be helpful in practical computation.

Theoretical analysis indicates that a good nodal array is of great importance to guarantee the design matrix with an acceptable restricted isometry constant (RIC) in the CS approach [23]. Rauhut and Ward [23, 24] investigated the recovery of expansions that are sparse in a univariate Legendre polynomial basis, and Yan *et al.* [35] extended their work to focus on the recoverability of stochastic solutions in high-dimensional random spaces, via ℓ_1-minimisation in a Legendre basis with random sampling. Hampton & Doostan [15] discussed convergence analysis and sampling strategies to recover a sparse stochastic function in both Hermite and Legendre gPC expansions for the ℓ_1-minimisation problem. Although random sampling methods have been used widely in the CS framework, a judicious deterministic choice of points may provide several advantages over randomly generated points. Xu & Zhou [34] presented deterministically constructed "Weil points" to recover sparse Chebyshev polynomials, and they removed the so called "failure probability". However, unlike the least-squares framework for UQ applications where low discrepancy points have been tested [4, 12, 20], stochastic collocation via CS by the Quasi-Monte Carlo method is rarely considered in the literature.

Here we emphasise the recoverability of stochastic solutions in high-dimensional random spaces, with evaluations at low discrepancy point sets. To facilitate practical implementation, we consider the case of uniform density and employ orthogonal multi-dimensional Legendre polynomials as basis functions. Random sets and the low discrepancy point sets, especially Halton and Sobol' sequences, are compared numerically in some case examples. The numerical tests show that both Halton and Sobol' sequences perform better than random sampling for low dimensional problems, and Halton sequences perform best. Furthermore, the QMC points and the random points show similar convergence properties in higher dimensions.