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On Preconditioned MHSS Real-Valued Iteration Methods for a Class of Complex Symmetric Indefinite Linear Systems

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Abstract. A generalized preconditioned modified Hermitian and skew-Hermitian splitting (GPMHSS) real-valued iteration method is proposed for a class of complex symmetric indefinite linear systems. Convergence theory is established and the spectral properties of an associated preconditioned matrix are analyzed. We also give several variants of the GPMHSS preconditioner and consider the spectral properties of the preconditioned matrices. Numerical examples illustrate the effectiveness of our proposed method.

AMS subject classifications: 65F10, 65F50

Key words: Complex linear systems, PMHSS iteration, real-valued form, convergence, preconditioning.

1. Introduction

We consider the iterative solution of a large, sparse and nonsingular linear system of the form

$$Ax = b\ , \ \text{ with } A = W + \mathrm{i}T\ ,\ x = y + \mathrm{i}z \ \text{ and } \ b = f + \mathrm{i}g\ , \tag{1.1}$$

where the $n \times n$ matrices W and T are real symmetric, the vectors y, z, f, g are all in $\mathbb{R}^n$, and $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. We assume that $W \neq 0$ and $T \neq 0$, which implies that the matrix A is neither Hermitian nor skew-Hermitian. Complex symmetric systems arise from many problems in scientific computing and engineering applications, including diffuse optical tomography [1], quantum mechanics [22], molecular scattering [20], structural dynamics [15], electrical power system modeling [17], lattice quantum chromodynamics [16], and so on; see also Ref. [10].

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Many efficient iteration methods have been proposed to solve the complex system (1.1). When W and T are symmetric matrices with one positive definite and the other either positive semidefinite or positive definite, Bai *et al.* [5, 6] introduced the modified Hermitian and skew-Hermitian splitting (MHSS) and preconditioned MHSS (PMHSS) iteration methods, which were proved more efficient than the HSS iteration method [9] for solving the complex linear system (1.1). They also established convergence theory for these MHSS and PMHSS iteration methods [5, 6]. In addition, the PMHSS method naturally involves a preconditioner for the complex symmetric matrix A, and some properties of the preconditioned matrix are also presented in Ref. [6]. When W and T are symmetric indefinite matrices and satisfy certain assumptions, Xu [24] presented a generalization of the PMHSS iteration method for the complex linear system (1.1). Recently, Cao & Ren [13] proposed two variants of the PMHSS iteration methods for a class of complex symmetric indefinite linear systems (1.1), and proved the convergence of these variants. However, the application of these methods to the complex system (1.1) requires complex arithmetic, which one may wish to avoid for several reasons. For instance, if most of the entries of A are real (with complex entries localized in just a few positions), then it is wasteful to use complex arithmetic throughout the code [12]. As mentioned in Ref. [14], another difficulty is the scarcity of available preconditioning software supporting complex arithmetic, compared to the widespread availability of high-quality packages that could be used for an equivalent real formulation of the complex linear system (1.1).

In fact, the complex linear system (1.1) can be transformed into the $2n \times 2n$ equivalent real formulation

$$\begin{pmatrix} W & -T \\ T & W \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{1.2}$$

The block two-by-two linear system (1.2) may be solved efficiently in real arithmetic by Krylov subspace methods such as the preconditioned GMRES and QMR [21], by C-to-R [2], or by alternating splitting iteration methods such as PHSS and PMHSS [7], for which high-quality preconditioners are crucial to guarantee their accuracy, efficiency and robustness in practical computation as discussed in Refs. [3, 10] and references therein. The C-to-R and the PMHSS iteration methods are presented for solving the system (1.2) when W and T are symmetric matrices with one positive definite and the other either positive semidefinite or positive definite. Moreover, the PMHSS iteration method [7] is a skilful preconditioned modification of the HSS iteration method initially introduced to solve non-Hermitian positive definite linear systems [5, 9]. It can be accommodated to solve the real linear system (1.2) using only real arithmetic. PMHSS iteration methods naturally result in matrix splitting preconditioners called PMHSS preconditioners, for Krylov subspace iteration methods such as GMRES and QMR employed to solve the complex and real linear systems (1.1) and (1.2). However, there are few effective real-valued iterative methods to solve the equivalent system (1.2) when the matrix W or the matrix T in the system (1.1) is symmetric indefinite.

In this paper, we explore preconditioned real-valued iteration methods for the complex symmetric indefinite linear system (1.1) with W and T satisfying the case $-T \prec W \preceq T$. Here and in the sequel, for any matrices B and C, the notation $B \prec C$ means that $C - B$