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Exponential Additive Runge-Kutta Methods for Semi-Linear Differential Equations

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Abstract. Exponential additive Runge-Kutta methods for solving semi-linear equations are discussed. Related order conditions and stability properties for both explicit and implicit schemes are developed, according to the dimension of the coefficients in the linear terms. Several examples illustrate our theoretical results.

AMS subject classifications: 65L20

Key words: Semi-linear differential equations, exponential Runge-Kutta methods, order conditions, stability.

1. Introduction

Recently, various semi-linear problems have emerged in many scientific fields such as physics, chemistry, biology, ecology and economics [5, 10, 11, 20]. Semi-linear differential equations are the most basic and important nonlinear evolution equations arising from the spatial discretisation of nonlinear time dependent partial differential equations (PDE) — cf. Refs. [8, 10]. There are already many results on the global existence of solutions for related initial boundary value and Cauchy problems — e.g. see Refs. [11, 24]. The numerical solution of initial value problems with systems of differential equations often but not always invoke discretisation in both space and time, and the exact solution may be approximated iteratively. In this article, we develop a new class of numerical methods for time integration of the following semi-linear problem in the time-dependent variable $u(t)$, an *a priori* partial linearisation of a nonlinear problem where the prime denotes the time derivative, L is a linear differential operator in the space variable, and $f(t,u(t))$ is a known function:

$$u'(t) = Lu(t) + f\big(t,u(t)\big)\,, \quad t \geq t_0\,, \tag{1.1}$$

subject to the initial condition $u(t_0) = u_0$.

Numerical methods for solving differential equations began to develop rapidly over fifty years ago, but more recently the construction of exponential integrators has provided

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exponential Runge-Kutta methods [12,13,18], exponential Rosenbrock methods [4,10,14], exponential general linear methods [19,22], exponential linear multistep methods [21], etc.. Exponential integration methods deal more efficiently with the linearity in semi-linear equations, and as their name suggests use the exponential function (and related functions) of the Jacobian or an approximation to it inside the related classical numerical method. In passing, we note that Ref. [15] handled the linear portions "exactly" by using integrating factors in formulating Krylov IIF-WENO methods for advection-diffusion-reaction systems, similar to exponential methods.

In many applications, it is convenient to use splitting methods to take advantage of the special structure of the differential operator in the mathematical model. The space discretisation of some time-dependent PDE produces ordinary differential equations that contain additive terms with different stiffness properties. In these situations, the additive Runge-Kutta methods have been used — e.g. see Refs. [1,2,6,17]. In molecular dynamics applications, the different structures may correspond to forces of different stiffness [2]. It is sometimes inappropriate to sample the net force, and one may wish to sample the stiffer parts more frequently than the non-stiff parts. For instance, in a reaction-diffusion partial differential problem, an implicit method for the diffusion may be combined with an explicit method for the reaction component [3,23]. A general approach is to construct implicit-explicit (IMEX) methods, where the stiff term is handled implicitly, and the non-stiff term is handled explicitly, thereby gaining the benefit of the implicit method for the stiffness but potentially saving computational effort by handling some terms explicitly. IMEX Runge-Kutta methods are also known as additive Runge-Kutta methods [6].

In Section 2, we establish exponential additive Runge-Kutta methods for semi-linear equations. The analysis of the order conditions for these methods is provided in Section 3, and stability properties are discussed in Section 4. We define exponentially algebraic stability and *EB*-stability; and then focus on exponentially additive Runge-Kutta methods dealing with equations with one-dimensional linear term coefficients, before considering the asymptotic stability of explicit exponential additive Runge-Kutta methods for equations with higher dimensional linear term coefficients. Section 5 is devoted to several examples that satisfy the order conditions, with numerical experiments to illustrate our theory. Some concluding remarks are made in Section 6.

2. Exponential Additive Runge-Kutta Methods

We consider the decomposition

$$f(t,u(t)) = \sum_{\nu=1}^{N} f_\nu\big(t,u(t)\big) \tag{2.1}$$

in Eq. (1.1), and exponential methods corresponding to non-additive counterparts. Let u_n denote the numerical approximation to the exact solution $u(t_0+nh)$ after n step sizes h. We obtain the general form of exponential additive Runge-Kutta methods on invoking a