East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.150216.260117a

Vol. **7**, No. 2, pp. 325-342
May 2017

Estimation of a Regularisation Parameter for a Robin Inverse Problem

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Received 15 February 2016; Accepted (in revised version) 26 January 2017.

Abstract. We consider the nonlinear and ill-posed inverse problem where the Robin coefficient in the Laplace equation is to be estimated using the measured data from the accessible part of the boundary. Two regularisation methods are considered — viz. L_2 and H^1 regularisation. The regularised problem is transformed to a nonlinear least squares problem; and a suitable regularisation parameter is chosen via the normalised cumulative periodogram (NCP) curve of the residual vector under the assumption of white noise, where information on the noise level is not required. Numerical results show that the proposed method is efficient and competitive.

AMS subject classifications: 65F22, 65R32

Key words: Robin inverse problem, L_2 regularisation, H^1 regularisation, normalised cumulative periodogram (NCP) method.

1. Introduction

We consider the Robin boundary value problem for the Laplace equation in a connected open domain $\Omega \subset R^2$:

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} + pu = g & \text{on} \quad \partial\Omega = \Gamma, \end{cases} \tag{1.1}$$

where Γ is the boundary of the domain Ω, ν is the unit outward normal direction on Γ, g is a given current input, and p is the Robin coefficient that is nonnegative and vanishes outside $\Gamma_1 \subset \Gamma$. The Robin inverse problem is to recover the Robin coefficient p by using the data $u_0 = u|_{\Gamma_0}$ where $\Gamma_0 \subset \Gamma$ and $\Gamma_0 \cap \Gamma_1 = \phi$. This problem arises in various nondestructive detection and thermal imaging investigations [1, 16], where an unknown material profile in a non-accessible part of the boundary Γ_1 is to be recovered from a partial boundary measurement made on an accessible part of the boundary Γ_0.

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In the last two decades, there has been considerable progress on numerical methods for the Robin inverse problem — cf. [3, 4, 10–12, 16–18, 20, 22, 23] and references therein. In order to recover the Robin coefficient numerically, we may either express p as a linear combination of certain basis functions [2, 3] or discretise the problem [10, 19]. One can discretise (1.1) directly using finite differences or finite elements (e.g. see Ref. [19]), or reformulate it as a boundary integral equation problem that is then discretised using the boundary element method or numerical quadratures (e.g. see Refs. [17, 22]). Indeed, since both the measured data u_0 and the unknown coefficient p are on the boundary Γ, it is natural to adopt boundary integral equation approach for the Robin inverse problem (e.g. see Refs. [5, 6, 8, 10, 22]). In particular, Fasino & Inglese [11, 12, 16] proposed efficient methods for the problem in the case where $\Omega = [0,1] \times [0,a]$, $\Gamma_1 = [0,1] \times \{a\}$, and $\Gamma_0 = [0,1] \times \{0\}$ by using a "thin-plate approximation"; Lin & Fang [22] proposed an efficient quadratic programming (QP) method, in which the nonlinear Robin inverse problem is transformed to a linear problem by introducing a new variable; Jin [17] transformed the problem into an optimisation problem, and then introduced two regularisation methods in conjunction with conjugate gradient (CG) methods; and Ma & Lin [23] proposed another CG method in the setting of the boundary integral equation and introduced a functional of p as a regularisation term. Here we further consider obtaining a regularisation parameter in the boundary integral equation setting for the problem (1.1), using measured data from the accessible part of the boundary.

The regularised problem may be represented in discrete form as a nonlinear least squares problem:

$$\min_{\mathbf{p}} \frac{1}{2}\left\|R(\mathbf{p})\right\|_2^2 + \frac{\mu}{2}\|W\mathbf{p}\|_2^2, \tag{1.2}$$

where $R(\mathbf{p})$ is the nonlinear function of $\mathbf{p}$ corresponding to the problem (1.1), W is a matrix corresponding to the regularisation method used, and $\mu > 0$ is a regularisation parameter. It is well known that the selection of a suitable value for the regularisation parameter μ is challenging, especially in the nonlinear context. Even for the linear case, there are many ways proposed in the literature — e.g. the discrepancy principle, the L-curve, the generalised cross-validation (GCV) and the normalised cumulative periodogram (NCP) curve. The discrepancy principle is based on the idea that the norm of the residual vector should be close to the norm of the noise in the measured data, which is known a priori [25]. The L-curve method plots the logarithm of the norm of the regularised solution against the squared norm of the corresponding residual for a range of values of the regularisation parameter, to choose the parameter value corresponding to the corner of the curve [15]. This is based on the fact that both the norms of the corresponding residual and the solution are functions of the regularisation parameter and that the norms have a special relation yielding the L-curve, with the parameter selected by seeking a point where the curvature is maximised. The NCP method gives the parameter value by checking if the corresponding residual vector is dominated by white noise [14]; and the GCV approach chooses the regularisation parameter by minimising the GCV function, which does not require any prior knowledge about the variance in the data [13].