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Semilocal Convergence Analysis for MMN-HSS Methods under Hölder Conditions

Yang Li and Xue-Ping Guo*

Department of Mathematics, East China Normal University, Shanghai 200241, China.

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Abstract. Multi-step modified Newton-HSS (MMN-HSS) methods, which are variants of inexact Newton methods, have been shown to be competitive for solving large sparse systems of nonlinear equations with positive definite Jacobian matrices. Previously, we established these MMN-HSS methods under Lipschitz conditions, and we now present a semilocal convergence theorem assuming the nonlinear operator satisfies milder Hölder continuity conditions. Some numerical examples demonstrate our theoretical analysis.

AMS subject classifications: 65F10, 65F50, 65H10

Key words: MMN-HSS method, large sparse systems of nonlinear equation, Hölder conditions, positive-definite Jacobian matrices, semilocal convergence.

1. Introduction

Systems of nonlinear equations arise in many practical applications, such as discretizations of nonlinear differential and integral equations, numerical optimization, see [3, 21, 27, 29] and references therein. We consider the general form

$$F(x) = 0\,, \tag{1.1}$$

where $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is a nonlinear continuously differentiable operator mapping from an open convex subset $\mathbb{D}$ of the n-dimensional complex linear space $\mathbb{C}^n$ into $\mathbb{C}^n$, and $F = (F_1, \cdots, F_n)^T$ with $F_i = F_i(x)$, $i = 1, 2, \cdots, n$ and $x = (x_1, \cdots, x_n)^T$ is such that the Jacobian matrix $F'(x)$ is large, sparse, non-Hermitian and positive definite.

Inexact Newton methods [17, 18, 22, 25] are especially useful for solving such systems, and the general algorithm can be summarised briefly as follows.

*Corresponding author. *Email addresses:* `jlqaplum@163.com` (Y. Li), `xpguo@math.ecnu.edu.cn` (X.-P. Guo)

Algorithm 1. IN (inexact Newton method)
1. Let x_0 be given.
2. For $k = 0, 1, \cdots$ until convergence do:
   2.1. For a given $\eta_k \in [0, 1)$, find s_k that satisfies

$$\left\| F(x_k) + F'(x_k)s_k \right\| \le \eta_k \left\| F(x_k) \right\|. \tag{1.2}$$

   2.2. Set

$$x_{k+1} = x_k + s_k .$$

Here $\eta_k \in [0, 1)$, commonly called the *forcing term*, is used to control the level of accuracy.

These methods are obviously variants of the classical Newton method where

$$F'(x_k)\, s_k = -F(x_k), \quad k \ge 0 \tag{1.3}$$

is solved approximately at each iteration. Local convergence analysis for the inexact Newton methods represented above likewise shows that the sequence $\{x_k\}$ converges if x_0 is sufficiently close to a solution x_* of the nonlinear system (1.1) and η_k is suitably uniformly bounded [17]. When the dimension n of the problem is large, linear iteration methods such as splitting methods and modern Krylov subspace methods [31] are usually applied to solve Eq. (1.3). Inexact Newton methods are inner-outer iteration methods, where the *inner iteration* invokes a linear method and a nonlinear *outer iteration* method generates the sequences $\{x_k\}$. The Newton-CG and Newton-GMRES methods, using CG and GMRES methods as the respective inner iterations, have often been used and studied [1, 13, 14]. For non-Hermitian positive-definite linear systems, Bai *et al.* [10] proposed the Hermitian and skew-Hermitian splitting (HSS) method, which converges unconditionally to the exact solution of the system of linear equations with the same upper bound for the convergence rate as the CG method when optimal parameters are used. Because of its effectiveness and robustness in successfully solving Stokes problems and distributed control problems for example [4, 5, 12], the HSS method has also been studied extensively [7–9, 11].

Using the HSS method as the inner iteration, Bai & Guo [6] proposed the Newton-HSS method for solving the system of nonlinear equations with non-Hermitian positive-definite Jacobian matrices.

Algorithm 2. NHSS (Newton-HSS method)
1. Given an initial guess x_0, positive constants α and tol, and a positive integer sequence $\{l_k\}_{k=0}^{\infty}$:
2. For $k = 0, 1, \cdots$ until $\|F(x_k)\| \le tol\|F(x_0)\|$ do:
   2.1. Set $d_{k,0} := 0$.
   2.2. For $l = 0, 1, \cdots, l_k - 1$, apply algorithm HSS:

$$\begin{cases} \big(\alpha I + H(x_k)\big)d_{k,l+\frac{1}{2}} = \big(\alpha I - S(x_k)\big)d_{k,l} - F(x_k), \\ \big(\alpha I + S(x_k)\big)d_{k,l+1} = \big(\alpha I - H(x_k)\big)d_{k,l+\frac{1}{2}} - F(x_k), \end{cases}$$