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A Chebyshev Spectral Collocation Method for Nonlinear Volterra Integral Equations with Vanishing Delays

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Abstract. A multistep Chebyshev-Gauss-Lobatto spectral collocation method for nonlinear Volterra integral equations with vanishing delays is developed. The convergence of the *hp*-version of the method in supremum norm is proved. Numerical experiments show the efficiency of the method for equations with highly oscillating, steep gradient and non-smooth solutions.

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Key words: Multistep Chebyshev-Gauss-Lobatto spectral collocation method, nonlinear Volterra integral equation, vanishing variable delay.

1. Introduction

Spectral methods have a number of advantages and are widely used in approximate solution of partial differential equations — cf. Refs. [5, 6, 13, 14, 17, 19, 25, 26]. Recently, spectral methods have been employed in initial value problems for ordinary differential equations (ODEs) and delay differential equations (DDEs). For example, Guo and Wang, along with other researchers, [20, 21, 31, 33, 37–40] developed Legendre spectral collocation methods for ODEs and DDEs, and Wang *et al.* [23,35] considered Chebyshev-Gauss and Chebyshev-Gauss-Lobatto spectral collocation methods for ODEs. In this work we investigate the Chebyshev-Gauss-Lobatto spectral collocation method for the following nonlinear

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Volterra integral equations (VIEs) with variable vanishing delays:

$$
\begin{aligned}
&y(t)=f(t)+\int_0^t K_1(t,s)G_1(s,y(s))ds+\int_0^{\theta(t)} K_2(t,s)G_2(s,y(s))ds,\\
&t\in I:=[0,T],
\end{aligned}
\tag{1.1}
$$

where $\theta(t):=t-\tau(t)$ is a delay function [9] such that

1. $\tau(0)=0,\ \tau(t)>0$ for $t\in(0,T]$ (vanishing delay);
2. $\theta(t)\in C^1(I)$ and $\theta'(t)\geq q_0>0$.

It is assumed that the kernels K_i, $i=1,2$ belong to the spaces $C(D_i)$ of continuous functions in the domains $D_1:=\{(t,s):\ 0\leq s\leq t\leq T\}$, $D_2:=\{(t,s):\ 0\leq s\leq \theta(t)\leq \theta(T)\}$, and the functions G_i are continuous on the corresponding subsets of $\mathbb{R}^2$ and $f\in C(I)$. Note that the Eq. (1.1) contains VIEs with proportional delays $\theta(t)=qt$, $0<q<1$ studied earlier in Refs. [7,15,24].

Numerical methods for Volterra integral and differential equations with delays have been also considered. For example, piecewise polynomial collocation for equations with vanishing delays are discussed in [4,7,9–12,34,36], while piecewise polynomial collocation methods and Runge-Kutta methods for equations with non-vanishing delay are studied in [7–9,30]. On the other hand, since spectral methods are very efficient in solving various problems with history, they become a popular tool in VIEs with delays — cf. [1–3,18,41]. However, the existing algorithms mainly comprise one-step methods, whereas for long time simulations, the multistep methods are more suitable. Several multistep methods have been considered recently: Conte and Paternoster [16] introduced a class of multistep collocation methods, each step of which uses Lagrange interpolation; Li *et al* [22] proposed a time parareal method with spectral-subdomain enhancement for VIEs; Sheng *et al.* [27] considered a multistep spectral collocation method for nonlinear VIEs and established the convergence of the *hp*-version; Wang *et al.* [28,32] developed a multistep Legendre-Gauss spectral collocation methods for nonlinear Volterra integral and Volterra functional integro-differential equations with delays.

In this work we develop an efficient multistep Chebyshev-Gauss-Lobatto spectral collocation method for Eq. (1.1). The main features of this work are:

- A multistep spectral collocation method for VIEs with nonlinear vanishing delays is considered. The existing works deal mainly with one-step methods with proportional or non-vanishing delays.

- Chebyshev expansions are used on each sub-step. The nodes and weights of Chebyshev-Gauss-Lobatto quadratures are given explicitly, which prevents a potential loss of accuracy. The algorithm can be implemented by exploiting the fast Chebyshev transform.

- The *hp*-convergence of the Chebyshev collocation method is fully analysed. A proper choice of h and p can significantly enhance the accuracy.