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Numerical Methods for Constrained Elliptic Optimal Control Problems with Rapidly Oscillating Coefficients

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Abstract. In this paper we use two numerical methods to solve constrained optimal control problems governed by elliptic equations with rapidly oscillating coefficients: one is finite element method and the other is multiscale finite element method. We derive the convergence analysis for those two methods. Analytical results show that finite element method can not work when the parameter ε is small enough, while multiscale finite element method is useful for any parameter ε.

AMS subject classifications: 65K05, 65K10, 65M60

Key words: Optimal control problems, finite element method, multiscale finite element method, homogenization, convergence analysis.

1. Introduction

Optimal control plays a very important role in many engineering applications. Efficient numerical methods are necessary to successful applications of optimal control. Finite element method seems to be the most widely used numerical method in computing optimal control problems, and the relevant literature is huge. It is impossible to give even a very brief review here. A systematic introduction of finite element method for PDEs and optimal control problems can be found in [1, 9, 10, 24, 26]. For elliptic and parabolic optimal control problems, a priori error estimates of finite element method were established in [18], a posteriori error estimates of residual type have been derived in [20, 21], a posteriori error estimates of recovery type have been derived in [17, 19], and some superconvergence results can be found in [2–4]. However, many fundament and practical problems in

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engineering have multiscale solutions, such as composite materials, porous media, turbulent transport in high Reynods number flows and so on. The direct numerical simulation of multiple scale problems is difficult even with modern supercomputers for the requisite of tremendous amount of computer memory and CPU time which can easily exceed the limitation of today's computer resources.

In practical applications, it is often sufficient to predict the large scale solutions to certain accuracy. Multiscale finite element method [8, 11, 13, 14, 27] provides an efficient way to capture the large scale structures of the solutions on a coarse mesh. The main idea is to construct multiscale finite element base functions which capture the local small scale information within each element. The small scale information is then brought to the large scales through the coupling of the global stiffness matrix. It is through these multiscale base functions and the finite element formulation that the effect of small scales on the large scales is correctly captured. Mixed multiscale finite element methods for multiscale problems can be found in [5, 15, 22]. Recently, Chu et al. investigated a new multiscale finite element method for high-contrast elliptic interface problems in [6] and Parvazinia considered a multiscale finite element for the solution of transport equations in [25].

The purpose of this work is to obtain the convergence analysis for finite element method and multiscale finite element method solving a constrained optimal control problems governed by elliptic equations with rapidly oscillating coefficients. Such problems often arise in composite materials and flows in porous media.

Let Ω be a bounded domain in $\mathbb{R}^n (n = 2, 3)$ with a Lipschitz boundary $\partial\Omega$. In this paper, we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and seminorm $|\cdot|_{W m,q(\Omega)}$. We set $H_0^1(\Omega) \equiv \left\{ v \in H^1(\Omega) : v|_{\partial\Omega} = 0 \right\}$ and denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$. In addition, c or C denotes a generic positive constant.

We are interested in the following optimal control problem:

$$\begin{cases} \min\limits_{u\in K} \left\{ \dfrac{1}{2}\|y^\varepsilon - y_d\|^2 + \dfrac{1}{2}\|u\|^2 \right\}, \\ -\nabla\cdot(A(x, x/\varepsilon)\nabla y^\varepsilon) = Bu, \quad \text{in } \Omega, \\ y^\varepsilon = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where K is a nonempty closed convex set in $L^2(\Omega)$, $A(x, x/\varepsilon)$ is a symmetric matrix which satisfies the uniform ellipticity condition:

$$\alpha|\xi|^2 \le a_{ij}(x, x/\varepsilon)\xi_i\xi_j \le \beta|\xi|^2, \ \forall \xi \in \mathbb{R}^n,$$

with $0 < \alpha < \beta$, $y_d \in L^2(\Omega)$, B is a continuous linear operator. Further more, we assume that $a_{ij}(x, \tilde{x})$ is periodic function with respect to the unit cube I in the "fast" variable $\tilde{x} = x/\varepsilon$, and

$$K = \left\{ v \in L^2(\Omega) \ : \ a \le v \le b, \ a.e. \text{ in } \Omega \right\},$$

where a and b are constants.

The paper is organized as follows: In Section 2, we shall construct a finite element approximation scheme and a multiscale finite element approximation scheme for the model