East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.251210.250411a

Vol. **1**, No. 3, pp. 248-263
August 2011

A Type of Finite Element Gradient Recovery Method based on Vertex-Edge-Face Interpolation: The Recovery Technique and Superconvergence Property

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Received 25 December 2010; Accepted (in revised version) 25 April 2011

Available online 27 July 2011

Abstract. In this paper, a new type of gradient recovery method based on vertex-edge-face interpolation is introduced and analyzed. This method gives a new way to recover gradient approximations and has the same simplicity, efficiency, and superconvergence properties as those of superconvergence patch recovery method and polynomial preserving recovery method. Here, we introduce the recovery technique and analyze its superconvergence properties. We also show a simple application in the a posteriori error estimates. Some numerical examples illustrate the effectiveness of this recovery method.

AMS subject classifications: 65N30, 65N12, 65N15, 65D10, 74S05

Key words: Finite element method, least-squares fitting, vertex-edge-face interpolation, superconvergence, a posteriori error estimate.

1. Introduction

Recently, a posteriori error estimates based on gradient recovery methods are active and attract more and more attention ([1–4, 8, 10, 12, 14–16, 20, 21, 23–25, 27]). One of the most widely used in practice is Zienkiewicz-Zhu's Superconvergence Patch Recovery (SPR) method ([27]) based on a local discrete least squares fitting. The popularity of this method relies on various factors: the method is rather independent of the problem, it is cheap to compute and easy to implement and the method works very well in practice. The robustness of the SPR method is dependent on its superconvergence property under structured meshes ([22]). However, [25] shows that the SPR is not superconvergence for linear element under the uniform triangulation of the Chevron pattern. The Polynomial Preserving Recovery (PPR) which overcomes this restriction is one of the most recent least-squares-based procedures ([16, 21, 24, 25]). This method is based on computing a local second order polynomial on a suitable patch associated with each mesh vertex via

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a discrete least-squares procedure. Then, the nodal gradient can be computed, which are family linearly interpolated. The PPR-recovered gradient has superconvergence properties in mildly structured meshes, and, in such cases, it was shown to be asymptotic exact ([21]). Both SPR and PPR select the node values as samples. The effectiveness of the gradient recovery method is rooted in the superconvergence theory. However, from the superconvergence theory ([12,14,15]), we know that the vertex-edge-face interpolations have better superconvergent properties than the common Lagrange interpolations. In this paper, a new type of gradient recovery method based on the vertex-edge-face interpolation is proposed and analyzed. The new gradient recovery method, given a finite element space of degree k, instead of gradient values at some sampling points on element patches (as in the SPR method and PPR method), selects gradient integration at the sampling edges and faces to obtain recovered gradient at each assembly vertex, edge and face. We shall prove that the new method has superconvergence for the superconvergent mesh (such as uniform triangular mesh of the Regular pattern and Chevron pattern, mildly meshes and so on) ([3,6,12,14,15,21,26]). In computer implementation, there is no significant difference between the new method with SPR or PPR compared with the overall cost in finite element solution.

The simple application of this recovery method to a posteriori error estimate is also discussed. The reader is referred to [1,2] for analysis of recovery type a posteriori error estimators.

The paper is organized as follows. We give the recovery technique in Section 2 and Section 3 is devoted to the superconvergence analysis. Section 4 shows the application of the recovery method to a posteriori error estimate. Numerical results are presented in Section 5. Finally, Section 6 contains some concluding remarks.

2. The Finite Element Method and Recovery Technique

This section is devoted to the introduction of the recovery technique. For simplicity, we consider the second order elliptic problem: Find a scalar function u such that

$$-\nabla\cdot(\mathscr{A}\nabla u)+bu=f, \qquad \text{in } \Omega, \tag{2.1}$$

$$u=u_D, \qquad \text{on } \partial\Omega, \tag{2.2}$$

where $\mathscr{A}\in\mathscr{R}^{2\times 2}$ is a positive definite matrix in Ω, $b\geq 0$ and $\Omega\subset\mathscr{R}^2$ is a bounded domain with Lipschitz boundary $\partial\Omega$.

In order to use the finite element method to compute the problem (2.1)-(2.2), we need to introduce a triangulation $\mathscr{T}_h$ on the domain Ω and then define the finite element space $S_h\subset H^1(\Omega)$ as

$$S_h=\big\{v\in H^1(\Omega): v|_e\in\mathscr{P}_k(e),\ \ \forall e\in\mathscr{T}_h\big\},$$

where $\mathscr{P}_k(e)$ is the space of polynomials of degree not greater than a positive integer k. The finite element method is to find $u_h\in S_h^D$ such that

$$a(u_h,v)=(f,v), \qquad \forall v\in S_h, \tag{2.3}$$