East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.281010.200411a

Vol. **1**, No. 3, pp. 284-296
August 2011

Mixed Fourier-Jacobi Spectral Method for Two-Dimensional Neumann Boundary Value Problems

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Received 28 October 2010; Accepted (in revised version) 20 April 2011

Available online 27 July 2011

Abstract. In this paper, we propose a mixed Fourier-Jacobi spectral method for two dimensional Neumann boundary value problem. This method differs from the classical spectral method. The homogeneous Neumann boundary condition is satisfied exactly. Moreover, a tridiagonal matrix is employed, instead of the full stiffness matrix encountered in the classical variational formulation. For analyzing the numerical error, we establish the mixed Fourier-Jacobi orthogonal approximation. The convergence of proposed scheme is proved. Numerical results demonstrate the efficiency of this approach.

AMS subject classifications: 33C45, 65M70, 35J25

Key words: Mixed Fourier-Jacobi orthogonal approximation, spectral method, Neumann boundary value problem.

1. Introduction

In the past several decades, spectral method has become increasingly popular in scientific computing and engineering applications (cf. [4–8, 13] and the references therein). In most of these applications, one usually considers spectral methods for Dirichlet boundary value problems. However, it is also important to consider various problems with Neumann boundary condition. In a standard variational formulation, this kind of boundary condition is commonly imposed in a natural way. Unfortunately, this approach usually leads to a full stiffness matrix for approximating the second derivatives.

To overcome this disadvantage, Shen [12] first introduced a Legendre spectral method with essential imposition of Neumann boundary condition. Moreover, Auteri et al. [2] also studied the aforementioned spectral solver for the Neumann problem associated with

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Laplace and Helmholtz operators in rectangular domains. This method differs from the classical spectral methods for such problems, the homogeneous Neumann boundary condition is satisfied exactly for each basis. In particular, the proposed approach leads to a diagonal stiffness matrix, rather than a full matrix encountered in the classical variational formulation. Wang and Wang [18] analyzed the numerical errors of this algorithm. Meanwhile, Yu and Wang [19] also developed Jacobi spectral method with essential imposition of Neumann boundary condition for one-dimensional Neumann boundary value problems.

In this paper, we investigate two-dimensional Neumann boundary value problem, using the Fourier-Jacobi spectral method with essential imposition of Neumann boundary condition. The main advantage of such treatment consists in that: (i). the stiffness matrix is tridiagonal, in contrast to the full stiffness matrix encountered in the classical variational formulation; (ii). the conservation of certain physical quantities can be retained for time-dependent problems. It is pointed out that Wang and Guo [15] also dealt with a heat transfer inside a unit disc with Dirichlet boundary condition, using Fourier-Jacobi spectral method.

For analyzing the numerical error, we establish basic result on mixed Fourier-Jacobi orthogonal approximation, motivated by Guo and Wang [10, 11], and Wang and Guo [16, 17]. The convergence of proposed scheme is proved. We also present some numerical results to demonstrate the efficiency of this approach.

This paper is organized as follows. In the next section, we recall some properties and relevant results of Jacobi approximations. The mixed Fourier-Jacobi orthogonal approximation for Neumann problem are established in Section 3. In Section 4, we propose the mixed Fourier-Jacobi spectral method with essential imposition of Neumann boundary condition for a model problem and analyze its numerical error. In Section 5, we present some numerical results. The final section is for concluding remarks.

2. Preliminaries

Let $\Lambda = \{x \mid |x| < 1\}$ and $\chi(x)$ be a certain weight function. Denote by $\mathbb{N}$ the set of all non-negative integers. For any $r \in \mathbb{N}$, we define the weighted Sobolev space $H^r_\chi(\Lambda)$ in the usual way, and denote its inner product, semi-norm and norm by $(u,v)_{r,\chi,\Lambda}$, $|v|_{r,\chi,\Lambda}$ and $\|v\|_{r,\chi,\Lambda}$ respectively. In particular, $L^2_\chi(\Lambda) = H^0_\chi(\Lambda)$, $(u,v)_{\chi,\Lambda} = (u,v)_{0,\chi,\Lambda}$ and $\|v\|_{\chi,\Lambda} = \|v\|_{0,\chi,\Lambda}$. For any $r > 0$, we define the space $H^r_\chi(\Lambda)$ by space interpolation as in [3]. In cases where no confusion arises, χ may be dropped from the notations whenever $\chi(x) \equiv 1$.

For $\alpha, \beta > -1$, we denote by $J^{(\alpha,\beta)}_l(x)$ the Jacobi polynomial of degree l, which is the eigenfunction of the following Sturm-Liouville problem

$$\partial_x((1-x)^{\alpha+1}(1+x)^{\beta+1}\partial_x v(x)) + \lambda^{(\alpha,\beta)}_l (1-x)^\alpha (1+x)^\beta v(x) = 0, \qquad x \in \Lambda, \tag{2.1}$$

with the corresponding eigenvalue $\lambda^{(\alpha,\beta)}_l = l(l+\alpha+\beta+1)$, $l \geq 0$. The Jacobi polynomials fulfill the following recurrence relations (cf. [1, 9, 14]),

$$\partial_x J^{(\alpha,\beta)}_l(x) = \frac{1}{2}(l+\alpha+\beta+1) J^{(\alpha+1,\beta+1)}_{l-1}(x), \qquad l \geq 1, \tag{2.2}$$

$$2(l+1)(l+\alpha+\beta+1)(2l+\alpha+\beta)J_{l+1}^{(\alpha,\beta)}(x)=\Big((2l+\alpha+\beta+1)(\alpha^2-\beta^2)$$
$$+\frac{(2l+\alpha+\beta+2)!}{(2l+\alpha+\beta-1)!}x\Big)J_l^{(\alpha,\beta)}(x)-2(l+\alpha)(l+\beta)(2l+\alpha+\beta+2)J_{l-1}^{(\alpha,\beta)}(x), \quad (2.3)$$

and

$$\int_{-1}^{x} J_l^{(\alpha,\beta)}(y)dy = a_l(J_{l+1}^{(\alpha,\beta)}(x)-J_{l+1}^{(\alpha,\beta)}(-1))+b_l(J_l^{(\alpha,\beta)}(x)-J_l^{(\alpha,\beta)}(-1))$$
$$+c_l(J_{l-1}^{(\alpha,\beta)}(x)-J_{l-1}^{(\alpha,\beta)}(-1)), \quad (2.4)$$

where

$$a_l=\frac{2(l+\alpha+\beta+1)}{(2l+\alpha+\beta+1)(2l+\alpha+\beta+2)}, \qquad b_l=\frac{2(\alpha-\beta)}{(2l+\alpha+\beta)(2l+\alpha+\beta+2)},$$
$$c_l=\frac{-2(l+\alpha)(l+\beta)}{(l+\alpha+\beta)(2l+\alpha+\beta)(2l+\alpha+\beta+1)}.$$

Besides,

$$J_l^{(\alpha,\beta)}(-x)=(-1)^l J_l^{(\beta,\alpha)}(x), \qquad J_l^{(\alpha,\beta)}(1)=\frac{\Gamma(l+\alpha+1)}{l!\Gamma(\alpha+1)}, \quad (2.5)$$

where $\Gamma(x)$ is the Gamma function.

Next let $\chi^{(\alpha,\beta)}(x)=(1-x)^\alpha(1+x)^\beta$. The set of Jacobi polynomials forms the $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$ orthogonal system,

$$\int_\Lambda J_l^{(\alpha,\beta)}(x)J_m^{(\alpha,\beta)}(x)\chi^{(\alpha,\beta)}(x)dx=\gamma_l^{(\alpha,\beta)}\delta_{l,m}, \quad (2.6)$$

where $\delta_{l,m}$ is the Kronecker function, and

$$\gamma_l^{(\alpha,\beta)}=\frac{2^{\alpha+\beta+1}\Gamma(l+\alpha+1)\Gamma(l+\beta+1)}{(2l+\alpha+\beta+1)\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}. \quad (2.7)$$

For any $N\in\mathbb{N}$, we denote by $\mathscr{P}_N$ the set of all algebraic polynomials of degree at most N. Let $\alpha,\beta,\gamma,\delta>-1$, we introduce the space $H^\mu_{\alpha,\beta,\gamma,\delta}(\Lambda)$, $0\le\mu\le1$ and $H^\mu_{\sigma,\lambda,\alpha,\beta,\gamma,\delta}(\Lambda)$, $0\le\mu\le2$. For $\mu=0$,

$$H^0_{\sigma,\lambda,\alpha,\beta,\gamma,\delta}(\Lambda)=H^0_{\alpha,\beta,\gamma,\delta}(\Lambda)=L^2_{\chi^{(\gamma,\delta)}}(\Lambda).$$

For $\mu=1$,

$$H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)=\left\{\, v \mid v \text{ is measurable and } \|v\|_{1,\alpha,\beta,\gamma,\delta,\Lambda}<\infty\right\},$$

equipped with the norm

$$\|v\|_{1,\alpha,\beta,\gamma,\delta,\Lambda}=\left(|v|^2_{1,\chi^{(\alpha,\beta)},\Lambda}+\|v\|^2_{\chi^{(\gamma,\delta)},\Lambda}\right)^{\frac12}.$$

For $\mu = 2$,

$$H^2_{\sigma,\lambda,\alpha,\beta,\gamma,\delta}(\Lambda) = \left\{ v \mid v \text{ is measurable and } \|v\|_{2,\sigma,\lambda,\alpha,\beta,\gamma,\delta,\Lambda} < \infty \right\},$$

equipped with the norm

$$\|v\|_{2,\sigma,\lambda,\alpha,\beta,\gamma,\delta,\Lambda} = \left(|v|^2_{2,\chi^{(\sigma,\lambda)},\Lambda} + |v|^2_{1,\chi^{(\alpha,\beta)},\Lambda} + \|v\|^2_{\chi^{(\gamma,\delta)},\Lambda} \right)^{\frac{1}{2}}.$$

The space $H^\mu_{\alpha,\beta,\gamma,\delta}(\Lambda)$, $0 < \mu < 1$ and $H^\mu_{\sigma,\lambda,\alpha,\beta,\gamma,\delta}(\Lambda)$, $0 < \mu < 2$ are defined by space interpolation as in [3], with the norms $\|v\|_{\mu,\alpha,\beta,\gamma,\delta,\Lambda}$ and $\|v\|_{\mu,\sigma,\lambda,\alpha,\beta,\gamma,\delta,\Lambda}$ respectively. For description of approximation results, we also define the space

$$H^r_{\chi^{(\alpha,\beta)},*}(\Lambda) = \left\{ v \mid v \text{ is measurable and } \|v\|_{r,\chi^{(\alpha,\beta)},*} < \infty \right\}, \qquad r \geq 1,\ r \in \mathbb{N},$$

where

$$\|v\|_{r,\chi^{(\alpha,\beta)},*} = \left(\sum_{k=0}^{r-1} |v|^2_{k+1,\chi^{(\alpha,\beta)},*} \right)^{\frac{1}{2}} \quad \text{and} \quad |v|_{r,\chi^{(\alpha,\beta)},*} = \|\partial_x^r v\|_{\chi^{(\alpha+r-1,\beta+r-1)},\Lambda}.$$

According to Lemma 3.5 of [9], one verifies readily that

Lemma 2.1. *If $\lambda < 1$, then for any $v \in H^2_{\sigma,\lambda,\alpha,\beta,\gamma,\delta}(\Lambda)$, $\partial_x v(x)$ is continuous on any subinterval $\Lambda^* = [-1, a] \subset \overline{\Lambda}$ with $-1 < a < 1$, and*

$$\max_{x \in \Lambda^*} |\partial_x v(x)| \leq c \|\partial_x v\|_{1,\chi^{(\sigma,\lambda)},\Lambda}.$$

If, in addition, $\sigma < 1$, then these results can be extended to $\overline{\Lambda}$.

In the forthcoming discussions, we need a unusual mapping. To do this, let $\lambda < 1$ and

$$\begin{aligned} {}^0H^2_{\sigma,\lambda,\alpha,\beta,\gamma,\delta}(\Lambda) &= \left\{ u \mid u \in H^2_{\sigma,\lambda,\alpha,\beta,\gamma,\delta}(\Lambda),\ \partial_x u(-1) = 0 \right\}, \\ {}^0\mathscr{P}_N(\Lambda) &= \mathscr{P}_N \cap {}^0H^2_{\sigma,\lambda,\alpha,\beta,\gamma,\delta}(\Lambda). \end{aligned}$$

Due to Lemma 2.1, the set ${}^0H^2_{\sigma,\lambda,\alpha,\beta,\gamma,\delta}(\Lambda)$ is meaningful.

Lemma 2.2. *(cf.* Theorem 3.3 *of* [19]*). If $\lambda < 1$ and one of the following conditions holds:*

$$\alpha \leq \gamma + 2,\ \alpha < 1,\ \beta \leq 0,\ \delta \geq 0, \tag{2.8}$$

$$\alpha \leq 0,\ \beta \leq \delta + 2,\ \gamma \geq 0, \tag{2.9}$$

$$\alpha \leq \gamma + 2,\ \beta \leq \delta + 1,\ \alpha < 1,\ 0 < \beta < 1, \tag{2.10}$$

then there exists a mapping

$${}^0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda} : {}^0H^2_{\sigma,\lambda,\alpha,\beta,\gamma,\delta}(\Lambda) \to {}^0\mathscr{P}_N(\Lambda),$$

such that ${}^0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}u(1) = u(1)$, *and for any* $u \in {}^0H^2_{\sigma,\lambda,\alpha,\beta,\gamma,\delta}(\Lambda) \cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ *with integer* $2 \le r \le N+1$,

$$\|{}^0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}u - u\|_{1,\alpha,\beta,\gamma,\delta,\Lambda} \le cN^{1-r}|u|_{r,\chi^{(\alpha,\beta)},*}. \tag{2.11}$$

In particular, if (2.8) *or* (2.10) *holds, then we have*

$$\|{}^0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}u - u\|_{\chi^{(-1,\delta)},\Lambda} \le cN^{1-r}|u|_{r,\chi^{(\alpha,\beta)},*}. \tag{2.12}$$

If, in addition,

$$0 < \alpha \le \gamma + 1 \text{ and } \frac{\lambda - 1}{2} \le \beta \le \delta + 1, \tag{2.13}$$

then for all $0 \le \mu \le 1$,

$$\|{}^0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}u - u\|_{\mu,\alpha,\beta,\gamma,\delta,\Lambda} \le cN^{\mu-r}|u|_{r,\chi^{(\alpha,\beta)},*}. \tag{2.14}$$

3. Mixed Fourier-Jacobi Orthogonal Approximation

In this section, we consider the mixed Fourier-Jacobi orthogonal approximation.

Let $I = (0, 2\pi)$ and $H^r(I)$ be the Sobolev space with the norm $\|\cdot\|_{r,I}$ and the semi-norm $|\cdot|_{r,I}$ as usual. For any non-negative integer m, we denote by $H^m_p(I)$ the subspace of $H^m(I)$, consisting of all functions whose derivatives of order up to $m-1$ have the period 2π. For any $r > 0$, the space $H^r_p(I)$ is defined by space interpolation as in [3].

Let M be any positive integer, and $\widetilde{V}_M(I) = \text{span}\{e^{il\theta} \mid |l| \le M\}$. We denote by $V_M(I)$ the subset of $\widetilde{V}_M(I)$ consisting of all real-valued functions. The orthogonal projection $P_{M,I}: L^2(I) \to V_M(I)$ is defined by

$$\int_I (P_{M,I}v(\theta) - v(\theta))\phi(\theta)d\theta = 0, \qquad \forall \phi \in V_M(I).$$

It was shown in [8] that for any $v \in H^r_p(I)$, $r \ge 0$ and $\mu \le r$,

$$\|P_{M,I}v - v\|_{\mu,I} \le cM^{\mu-r}|v|_{r,I}. \tag{3.1}$$

We now establish the result on the mixed Fourier-Jacobi orthogonal approximation. For this purpose, let $\Omega = \Lambda \times I$. We define the spaces

$$\begin{aligned}\mathscr{F}(\Omega) &:= \mathscr{F}(\sigma,\lambda,\alpha,\beta,\gamma,\delta,\eta,\xi) = \Big\{v \in H^2_{\sigma,\lambda,\alpha,\beta,\gamma,\delta}(\Lambda, H^1_p(I)) \mid \text{there exists finite trace of} \\ &\qquad \partial_x v(x,\theta) \text{ at } x=-1 \text{ and } \|v\|_{1,\alpha,\beta,\gamma,\delta,\eta,\xi,\Omega} < \infty\Big\}, \\ {}^0\mathscr{F}(\Omega) &:= {}^0\mathscr{F}(\sigma,\lambda,\alpha,\beta,\gamma,\delta,\eta,\xi) = \{v \in \mathscr{F}(\Omega) \mid \partial_x v(-1,\theta) = 0\},\end{aligned}$$

where

$$\|v\|_{1,\alpha,\beta,\gamma,\delta,\eta,\xi,\Omega} = \left(\|\partial_x v\|^2_{L^2_{\chi^{(\alpha,\beta)}}(\Lambda,L^2(I))} + \|\partial_\theta v\|^2_{L^2_{\chi^{(\eta,\xi)}}(\Lambda,L^2(I))} + \|v\|^2_{L^2_{\chi^{(\gamma,\delta)}}(\Lambda,L^2(I))}\right)^{\frac{1}{2}}.$$

Moreover, we denote by

$$(u,v)_{\chi,\Omega}=\int_\Omega u(x,\theta)v(x,\theta)\chi(x)d\theta dx.$$

Next denote by $\mathscr{P}_{N,M}(\Omega)=\mathscr{P}_N(\Lambda)\otimes V_M(I)\cap {}^0\mathscr{F}(\Omega)$. The orthogonal projection ${}^0P^1_{N,M,\Omega}$: ${}^0\mathscr{F}(\Omega)\to\mathscr{P}_{N,M}(\Omega)$ is defined by

$$a({}^0P^1_{N,M,\Omega}v-v,\phi)=0,\qquad \forall\phi\in\mathscr{P}_{N,M}(\Omega), \tag{3.2}$$

where

$$a(u,v)=(\partial_x u,\partial_x v)_{\chi^{(\alpha,\beta)},\Omega}+(\partial_\theta u,\partial_\theta v)_{\chi^{(\eta,\xi)},\Omega}+(u,v)_{\chi^{(\gamma,\delta)},\Omega}.$$

Clearly, $\mathscr{P}_{N,M}(\Omega)$ and ${}^0P^1_{N,M,\Omega}$ are related to the parameters $\sigma,\lambda,\alpha,\beta,\gamma,\delta,\eta,\xi$.

Lemma 3.1. *For any $v(\cdot,\theta)\in L^2(I)$ and $\partial_\theta v(1,\theta)=0$, we have $\partial_\theta P_{M,I}v(1,\theta)=0$.*

Proof. Due to $\partial_\theta v(1,\theta)=0$, we can rewrite $v(x,\theta)$ as $v(x,\theta)=(1-x)^\mu u(x,\theta)+b(x)$, where $\mu>0$ is a certain constant. Thanks to $v(\cdot,\theta)\in L^2(I)$, we deduce readily that $u(\cdot,\theta)\in L^2(I)$. Hence $P_{M,I}u(x,\theta)$ is meaningful. Furthermore, $P_{M,I}v(x,\theta)=(1-x)^\mu P_{M,I}u(x,\theta)+b(x)$. Hence, $\partial_\theta P_{M,I}v(1,\theta)=0$. ☐

Theorem 3.1. *(i). If one of the conditions (2.8)-(2.10) holds, then for any*

$$v\in {}^0\mathscr{F}(\Omega)\cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda,H^1(I))\cap H^1_{\alpha,\beta,\gamma,\delta}(\Lambda,H^s_p(I))$$

with $\alpha,\beta,\gamma,\delta,\eta,\xi>-1$, integer $2\le r\le N+1$, $s\ge 1$, $\eta\ge\gamma$ and $\xi\ge\delta$, we have

$$\begin{aligned}\|{}^0P^1_{N,M,\Omega}v-v\|_{1,\alpha,\beta,\gamma,\delta,\eta,\xi,\Omega}\le & c(N^{1-r}+M^{1-s})\Big(|v|_{H^r_{\chi^{(\alpha,\beta)},*}(\Lambda,L^2(I))}+|v|_{H^r_{\chi^{(\alpha,\beta)},*}(\Lambda,H^1(I))}\\ &+|\partial_x v|_{L^2_{\chi^{(\alpha,\beta)}}(\Lambda,H^s(I))}+|v|_{L^2_{\chi^{(\gamma,\delta)}}(\Lambda,H^s(I))}\Big).\end{aligned} \tag{3.3}$$

(ii). If $\eta=-1$, (2.8) or (2.10) holds and $\partial_\theta v(1,\theta)=0$, then for any

$$v\in {}^0\mathscr{F}(\Omega)\cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda,H^1(I))\cap H^1_{\alpha,\beta,\gamma,\delta}(\Lambda,H^s_p(I))\cap L^2_{\chi^{(-1,\delta)}}(\Lambda,H^s(I))$$

with $\alpha,\beta,\gamma,\delta,\xi>-1$, integer $2\le r\le N+1$, $s\ge 1$ and $\xi\ge\delta$, we have

$$\begin{aligned}\|{}^0P^1_{N,M,\Omega}v-v\|_{1,\alpha,\beta,\gamma,\delta,-1,\xi,\Omega}\le & c(N^{1-r}+M^{1-s})\Big(|v|_{H^r_{\chi^{(\alpha,\beta)},*}(\Lambda,L^2(I))}+|v|_{H^r_{\chi^{(\alpha,\beta)},*}(\Lambda,H^1(I))}\\ &+|\partial_x v|_{L^2_{\chi^{(\alpha,\beta)}}(\Lambda,H^s(I))}+|v|_{L^2_{\chi^{(-1,\delta)}}(\Lambda,H^s(I))}\Big).\end{aligned} \tag{3.4}$$

Proof. We first consider the case (3.3). By the projection theorem, we have

$$\|{}^0P^1_{N,M,\Omega}v-v\|_{1,\alpha,\beta,\gamma,\delta,\eta,\xi,\Omega}\le\|\phi-v\|_{1,\alpha,\beta,\gamma,\delta,\eta,\xi,\Omega},\quad\forall\phi\in\mathscr{P}_{N,M}(\Omega). \tag{3.5}$$

Take $\phi = {}^0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}\cdot P_{M,I}v$. Since $\alpha,\beta,\gamma,\delta,\eta,\xi > -1$, we verify readily that $\phi \in \mathscr{P}_{N,M}(\Omega)$. It remains to estimate the terms $\|{}^0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}\cdot P_{M,I}v - v\|_{H^1_{\alpha,\beta,\gamma,\delta}(\Lambda,L^2(I))}$ and $\|\partial_\theta({}^0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}\cdot P_{M,I}v - v)\|_{L^2_{\chi^{(\eta,\xi)}}(\Lambda,L^2(I))}$. Thanks to (2.11) and (3.1), we deduce that for integer $2\le r\le N+1$ and $s\ge 0$,

$$\begin{aligned}
&\|{}^0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}\cdot P_{M,I}v - v\|_{H^1_{\alpha,\beta,\gamma,\delta}(\Lambda,L^2(I))}\\
&\le \|{}^0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}\cdot P_{M,I}v - P_{M,I}v\|_{H^1_{\alpha,\beta,\gamma,\delta}(\Lambda,L^2(I))} + \|P_{M,I}v - v\|_{H^1_{\alpha,\beta,\gamma,\delta}(\Lambda,L^2(I))}\\
&\le cN^{1-r}|P_{M,I}v|_{H^r_{\chi^{(\alpha,\beta)},*}(\Lambda,L^2(I))} + cM^{-s}|\partial_x v|_{L^2_{\chi^{(\alpha,\beta)}}(\Lambda,H^s(I))} + cM^{-s}|v|_{L^2_{\chi^{(\gamma,\delta)}}(\Lambda,H^s(I))}\\
&\le cN^{1-r}|v|_{H^r_{\chi^{(\alpha,\beta)},*}(\Lambda,L^2(I))} + cM^{-s}|\partial_x v|_{L^2_{\chi^{(\alpha,\beta)}}(\Lambda,H^s(I))} + cM^{-s}|v|_{L^2_{\chi^{(\gamma,\delta)}}(\Lambda,H^s(I))}.
\end{aligned}\tag{3.6}$$

Moveover, due to $\eta\ge\gamma$ and $\xi\ge\delta$, we use (2.11) and (3.1) again to obtain that for integer $2\le r\le N+1$ and $s\ge 1$,

$$\begin{aligned}
&\|\partial_\theta({}^0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}\cdot P_{M,I}v - v)\|_{L^2_{\chi^{(\eta,\xi)}}(\Lambda,L^2(I))}\\
&\le \|{}^0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}\cdot \partial_\theta P_{M,I}v - \partial_\theta P_{M,I}v\|_{L^2_{\chi^{(\gamma,\delta)}}(\Lambda,L^2(I))} + \|\partial_\theta(P_{M,I}v - v)\|_{L^2_{\chi^{(\gamma,\delta)}}(\Lambda,L^2(I))}\\
&\le cN^{1-r}|\partial_\theta P_{M,I}v|_{H^r_{\chi^{(\alpha,\beta)},*}(\Lambda,L^2(I))} + cM^{1-s}|v|_{L^2_{\chi^{(\gamma,\delta)}}(\Lambda,H^s(I))}\\
&\le cN^{1-r}|v|_{H^r_{\chi^{(\alpha,\beta)},*}(\Lambda,H^1(I))} + cM^{1-s}|v|_{L^2_{\chi^{(\gamma,\delta)}}(\Lambda,H^s(I))}.
\end{aligned}\tag{3.7}$$

Therefore, a combination of (3.6) and (3.7) leads to (3.3).

Next if $\eta=-1$, (2.8) or (2.10) holds and $\partial_\theta v(1,\theta)=0$, then we take $\phi = {}^0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}\cdot P_{M,I}v$. Since $\partial_\theta\phi = {}^0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}\partial_\theta P_{M,I}v$. Moreover, according to Lemma 2.2, ${}^0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}u(1) = u(1)$. Therefore, by virtue of Lemma 3.1, $\partial_\theta\phi(1,\theta) = \partial_\theta P_{M,I}v(1,\theta) = 0$, and so $\partial_\theta\phi(x,\cdot)\in L^2_{\chi^{(-1,\xi)}}(\Lambda)$. This leads to $\phi\in\mathscr{P}_{N,M}(\Omega)$. It remains to estimate the term $\|\partial_\theta({}^0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}\cdot P_{M,I}v - v)\|_{L^2_{\chi^{(-1,\xi)}}(\Lambda,L^2(I))}$. Thanks to $\xi\ge\delta$, we obtain from (2.12) and (3.1) that

$$\begin{aligned}
&\|\partial_\theta({}^0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}\cdot P_{M,I}v - v)\|_{L^2_{\chi^{(-1,\xi)}}(\Lambda,L^2(I))}\\
&\le \|{}^0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}\cdot \partial_\theta P_{M,I}v - \partial_\theta P_{M,I}v\|_{L^2_{\chi^{(-1,\delta)}}(\Lambda,L^2(I))} + \|\partial_\theta(P_{M,I}v - v)\|_{L^2_{\chi^{(-1,\delta)}}(\Lambda,L^2(I))}\\
&\le cN^{1-r}|\partial_\theta P_{M,I}v|_{H^r_{\chi^{(\alpha,\beta)},*}(\Lambda,L^2(I))} + cM^{1-s}|v|_{L^2_{\chi^{(-1,\delta)}}(\Lambda,H^s(I))}\\
&\le cN^{1-r}|v|_{H^r_{\chi^{(\alpha,\beta)},*}(\Lambda,H^1(I))} + cM^{1-s}|v|_{L^2_{\chi^{(-1,\delta)}}(\Lambda,H^s(I))}.
\end{aligned}\tag{3.8}$$

Therefore, a combination of (3.6) and (3.8) leads to (3.4). □

4. Mixed Fourier-Jacobi Spectral Method for Neumann Problem

In this section, we investigate the mixed spectral method with essential imposition of Neumann boundary condition for two-dimensional problem. For simplicity, we consider the following model problem

$$\begin{cases} -\Delta V(y_1,y_2)+\mu V(y_1,y_2)=G(y_1,y_2), & \mu>0, \quad y_1^2+y_2^2<2, \\ -\partial_n V(y_1,y_2)=0, & y_1^2+y_2^2=2. \end{cases} \tag{4.1}$$

Let $y_1=\rho\cos\theta$, $y_2=\rho\sin\theta$, $W(\rho,\theta)=V(y_1,y_2)$ and $F(\rho,\theta)=G(y_1,y_2)$. Then the above equation can be rewritten in polar coordinates as

$$\begin{cases} -\frac{1}{\rho}\partial_\rho(\rho\partial_\rho W(\rho,\theta))-\frac{1}{\rho^2}\partial_\theta^2 W(\rho,\theta)+\mu W(\rho,\theta)=f(\rho,\theta), & 0\le\rho<2, 0\le\theta<2\pi, \\ W(\rho,\theta+2\pi)=W(\rho,\theta), & 0\le\rho<2, 0\le\theta<2\pi, \\ \partial_\rho W(2,\theta)=0, & 0\le\theta<2\pi. \end{cases} \tag{4.2}$$

Moreover, we have the polar condition $\partial_\theta U(0,\theta)=0$ for $0\le\theta<2\pi$. We make the variable transformation $\rho=1-x$, $U(x,\theta)=W(\rho,\theta)$, $f(x,\theta)=F(\rho,\theta)$. Then (4.2) can be changed to

$$\begin{cases} -\frac{1}{1-x}\partial_x((1-x)\partial_x U(x,\theta))-\frac{1}{(1-x)^2}\partial_\theta^2 U(x,\theta)+\mu U(x,\theta)=f(x,\theta), & \text{in } \Omega, \\ U(x,\theta+2\pi)=U(x,\theta), & \text{in } \Omega, \\ \partial_x U(-1,\theta)=0, \quad \partial_\theta U(1,\theta)=0, & 0\le\theta<2\pi. \end{cases} \tag{4.3}$$

In order to derive a proper weak formulation of (4.3), we introduce the bilinear form with $\mu>0$,

$$\begin{aligned} b_\mu(u,v)=&\int_\Omega (1-x)\partial_x u(x,\theta)\partial_x v(x,\theta)d\theta dx+\int_\Omega \frac{1}{1-x}\partial_\theta u(x,\theta)\partial_\theta v(x,\theta)d\theta dx \\ &+\mu\int_\Omega (1-x)u(x,\theta)v(x,\theta)d\theta dx. \end{aligned} \tag{4.4}$$

In the forthcoming discussions, let $\|\cdot\|_{1,A}=\|\cdot\|_{1,1,0,1,0,-1,0,\Omega}$, and still denote by ${}^0\mathscr{F}(\Omega)$, ${}^0P^1_{N,M,\Omega}$ and $\mathscr{P}_{N,M}(\Omega)$ the corresponding notations as before with $\alpha=\gamma=1$, $\beta=\delta=\xi=0$ and $\eta=-1$. Due to $\partial_\theta u(1,\theta)=0$, we get that

$$|b_\mu(u,v)|\le\max(\mu,1)\|u\|_{1,A}\|v\|_{1,A}, \qquad b_\mu(u,u)\ge\min(\mu,1)\|u\|^2_{1,A}. \tag{4.5}$$

The weak formulation of (4.3) is to find $U\in{}^0\mathscr{F}(\Omega)$ such that

$$b_\mu(U,v)=(f,v)_\Omega, \qquad \forall v\in{}^0\mathscr{F}(\Omega). \tag{4.6}$$

If $f\in{}^0\mathscr{F}(\Omega)'$, then by (4.5) and the Lax-Milgram lemma, (4.6) admits a unique solution.

The mixed spectral scheme for (4.6) is to seek $u_{N,M}\in\mathscr{P}_{N,M}(\Omega)$ such that

$$b_\mu(u_{N,M},\phi)=(f,\phi)_\Omega, \qquad \forall\phi\in\mathscr{P}_{N,M}(\Omega). \tag{4.7}$$

Theorem 4.1. *If*

$$U \in {}^0\mathscr{F}(\Omega) \cap H^r_{\chi^{(1,0)},*}(\Lambda, H^1(I)) \cap H^1_{1,0,1,0}(\Lambda, H^s_p(I)) \cap L^2_{\chi^{(-1,0)}}(\Lambda, H^s(I)),$$

then for integer $2 \le r \le N+1$ *and* $s \ge 1$,

$$\begin{aligned}\|U - u_{N,M}\|_{1,A} \le & c(N^{1-r} + M^{1-s})\Big(|v|_{H^r_{\chi^{(1,0)},*}(\Lambda, L^2(I))} + |v|_{H^r_{\chi^{(1,0)},*}(\Lambda, H^1(I))} \\ & + |\partial_x v|_{L^2_{\chi^{(1,0)}}(\Lambda, H^s(I))} + |v|_{L^2_{\chi^{(-1,0)}}(\Lambda, H^s(I))}\Big).\end{aligned}$$

Proof. Let $U_{N,M} = {}^0P^1_{N,M,\Omega}U$. By the definition (3.2), we obtain from (4.6) that

$$b_\mu(U_{N,M}, \phi) = (\mu - 1)(U_{N,M} - U, \phi)_{\chi^{(1,0)},\Omega} + (f, \phi)_\Omega, \quad \forall \phi \in \mathscr{P}_{N,M}(\Omega). \tag{4.8}$$

Further, let $\tilde{u}_{N,M} = u_{N,M} - U_{N,M}$. Subtracting (4.8) from (4.7) yields

$$b_\mu(\tilde{u}_{N,M}, \phi) = (\mu - 1)(U - U_{N,M}, \phi)_{\chi^{(1,0)},\Omega}.$$

Taking $\phi = \tilde{u}_{N,M}$, we use (4.5) to assert that

$$\|\tilde{u}_{N,M}\|^2_{1,A} \le c\|U - U_{N,M}\|_{\chi^{(1,0)},\Omega}\|\tilde{u}_{N,M}\|_{\chi^{(1,0)},\Omega}.$$

Hence

$$\|\tilde{u}_{N,M}\|_{1,A} \le c\|U - U_{N,M}\|_{\chi^{(1,0)},\Omega}.$$

This fact with (3.4) leads to the desired result. □

5. Numerical Results

In this section, we describe the numerical implementations and present some numerical results confirming the theoretical analysis in the last section.

Denote by $L_k(x)$ the Legendre polynomial of degree k, and set

$$\begin{aligned}\psi_k(x) =& L_k(x) - \frac{2k+3}{(k+2)^2}L_{k+1}(x) - \frac{(k+1)^2}{(k+2)^2}L_{k+2}(x) \\ =& (1-x)(J^{(1,0)}_k(x) + \frac{(k+1)^2}{(k+2)^2}J^{(1,0)}_{k+1}(x)), \qquad 0 \le k \le N-2.\end{aligned}$$

Clearly, $\partial_x\psi_k(-1) = 0$ and $\psi_k(1) = 0$, $0 \le k \le N-2$. Moreover, let

$$\varphi_0(x) = L_0(x), \quad \varphi_k(x) = \sqrt{\frac{k+2}{2k(k+1)}}(L_k(x) + \frac{k}{k+2}L_{k+1}(x)), \quad 1 \le k \le N-1.$$

Then we have $\partial_x \varphi_k(-1) = 0$, $0 \le k \le N-1$. We now take the basis functions as

$$\begin{cases} \phi^1_{k,m}(x,\theta) = \dfrac{1}{\sqrt{2\pi}} \psi_k(x) \sin(m\theta), & 0 \le k \le N-2, \quad 1 \le m \le M, \\ \phi^2_{k,m}(x,\theta) = \dfrac{1}{\sqrt{2\pi}} \psi_k(x) \cos(m\theta), & 0 \le k \le N-2, \quad 1 \le m \le M, \\ \phi^3_{k}(x,\theta) = \dfrac{1}{\sqrt{2\pi}} \varphi_k(x), & 0 \le k \le N-1. \end{cases}$$

It can be checked readily that $\partial_x \phi^q_{k,m}(-1,\theta) = 0$, $\partial_x \phi^3_k(-1,\theta) = 0$, $\partial_\theta \phi^q_{k,m}(1,\theta) = 0$ and $\partial_\theta \phi^3_k(1,\theta) = 0$, $q = 1,2$. In particular, the set of the previous basis functions spans the space $\mathscr{P}_{N,M}(\Omega)$. The numerical solution is expanded as

$$u_{N,M}(x,\theta) = \sum_{k=0}^{N-2} \sum_{m=1}^{M} \widehat{u}^1_{k,m} \phi^1_{k,m}(x,\theta) + \sum_{k=0}^{N-2} \sum_{m=1}^{M} \widehat{u}^2_{k,m} \phi^2_{k,m}(x,\theta) + \sum_{k=0}^{N-1} \widehat{u}^3_k \phi^3_k(x,\theta).$$

Next take $\phi = \phi^q_{j,l}(x,\theta)$ and $\phi = \phi^3_j(x,\theta)$ in (4.7), and let $f^q_{j,l} = \int_\Omega f(x,\theta)\phi^q_{j,l}(x,\theta)d\theta dx$ and $f^3_j = \int_\Omega f(x,\theta)\phi^3_j(x,\theta)d\theta dx$. Then by the orthogonality of trigonometric functions, we deduce that

$$\begin{cases} \displaystyle\sum_{k=0}^{N-2} \bigg(\bigg(\int_\Lambda (1-x)\partial_x \psi_k \partial_x \psi_j dx + l^2 \int_\Lambda \frac{1}{1-x} \psi_k \psi_j dx \\ \displaystyle\quad + \mu \int_\Lambda (1-x) \psi_k \psi_j dx \bigg) \widehat{u}^q_{k,l} = 2 f^q_{j,l}, \qquad 0 \le j \le N-2, \ 1 \le l \le M, \ q = 1,2, \\ \displaystyle\sum_{k=0}^{N-1} \bigg(\int_\Lambda (1-x)\partial_x \varphi_k \partial_x \varphi_j dx + \mu \int_\Lambda (1-x) \varphi_k \varphi_j dx \bigg) \widehat{u}^3_k = f^3_j, \qquad 0 \le j \le N-1. \end{cases} \tag{5.1}$$

For deriving a compact matrix of the above equations, we introduce the matrices $\mathbb{A} = (a_{j,k})$, $\mathbb{B} = (b_{j,k})$, $\mathbb{C} = (c_{j,k})$, $0 \le j,k \le N-2$ and $\mathbb{G} = (g_{j,k})$, $\mathbb{H} = (h_{j,k})$, $0 \le j,k \le N-1$ with the following entries:

$$\begin{aligned} a_{j,k} &= \int_{-1}^{1} (1-x)\partial_x \psi_k(x) \partial_x \psi_j(x) dx, & b_{j,k} &= \int_{-1}^{1} \frac{1}{1-x} \psi_k(x) \psi_j(x) dx, \\ c_{j,k} &= \int_{-1}^{1} (1-x) \psi_k(x) \psi_j(x) dx, & g_{j,k} &= \int_{-1}^{1} (1-x)\partial_x \varphi_k(x) \partial_x \varphi_j(x) dx, \\ h_{j,k} &= \int_{-1}^{1} (1-x) \varphi_k(x) \varphi_j(x) dx. \end{aligned}$$

We next calculate the non zero elements of the matrices $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$. By using (2.6) we obtain that for $0 \le k \le N-2$,

$$a_{kk} = \frac{2(k+1)(2k+3)(k^2+3k+1)}{(k+2)^3}, \qquad a_{k(k+1)} = a_{(k+1)k} = -\frac{2(k+1)^2}{k+2},$$
$$b_{kk} = \frac{2(k+1)^5+2(k+2)^5}{(k+1)(k+2)^5}, \qquad b_{k(k+1)} = b_{(k+1)k} = \frac{2(k+1)^2}{(k+2)^3},$$
$$c_{kk} = \frac{4(k+1)(2k+3)(k^2+3k+11)}{(k+2)^3(2k+1)(2k+5)},$$
$$c_{k(k+1)} = c_{(k+1)k} = -\frac{2(k^4+8k^3+52k^2+144k+74)}{(k+2)(k+3)^2(2k+1)(2k+7)},$$
$$c_{k(k+2)} = c_{(k+2)k} = -\frac{2(k^4+10k^3+28k^2+15k-13)}{(k+2)^2(k+4)^2(2k+5)},$$
$$c_{k(k+3)} = c_{(k+3)k} = \frac{2(k+1)^2(k+3)}{(k+2)^2(2k+5)(2k+7)}.$$

Similarly, the non zero elements of the matrices $\mathbb{G}$ and $\mathbb{H}$ for $1 \le k \le N-1$ are as follows,

$$g_{kk} = 1, \qquad h_{kk} = \frac{2(k^2+2k+6)}{k(k+2)(2k+1)(2k+3)},$$
$$h_{k(k+1)} = h_{(k+1)k} = -\frac{6(3k^2+9k+5)\sqrt{k(k+3)}}{k(k+1)(k+2)(k+3)(2k+1)(2k+3)(2k+5)},$$
$$h_{k(k+2)} = h_{(k+2)k} = -\frac{\sqrt{k(k+1)(k+3)(k+4)}}{(k+1)(k+3)(2k+3)(2k+5)}.$$

In particular, $h_{00} = 2,\ h_{01} = h_{10} = -\sqrt{3}/3$. Next let

$$X_l^q = (\widehat{u}_{0,l}^q, \widehat{u}_{1,l}^q, \cdots, \widehat{u}_{N-2,l}^q)^T, \qquad F_l^q = (f_{0,l}^q, f_{1,l}^q, \cdots, f_{N-2,l}^q)^T, \quad 1 \le l \le M, \quad q = 1,2,$$
$$X^3 = (\widehat{u}_0^3, \widehat{u}_1^3, \cdots, \widehat{u}_{N-1}^3)^T, \qquad F^3 = (f_0^3, f_1^3, \cdots, f_{N-1}^3)^T.$$

Then we have from (5.1) that

$$[\mathbb{A} + l^2\mathbb{B} + \mu\mathbb{C}]X_l^q = 2F_l^q, \quad 1 \le l \le M, \quad q = 1,2, \tag{5.2}$$
$$[\mathbb{G} + \mu\mathbb{H}]X^3 = 2F^3. \tag{5.3}$$

For description of the numerical errors, let $\theta_{M,l} = 2\pi l/(2M+1)$, $0 \le l \le 2M$, and $\zeta_{N,k}$ and $\rho_{N,k}$, $0 \le k \le N$ be the zeros and weights of Legendre-Gauss interpolation,

$$E_{M,N,1} = \left(\frac{2\pi}{2M+1} \sum_{k=0}^{N} \sum_{l=0}^{2M} (U(\zeta_{N,k}, \theta_{M,l}) - u_{M,N}(\zeta_{N,k}, \theta_{M,l}))^2 \rho_{N,k} \right)^{\frac{1}{2}} \simeq \left\| U - u_{M,N} \right\|_{L^2(\Omega)},$$
$$E_{M,N,2} = \max_{0 \le k \le N} \max_{0 \le l \le 2M} \left| U(\zeta_{N,k}, \theta_{M,l}) - u_{M,N}(\zeta_{N,k}, \theta_{M,l}) \right| \simeq \left\| U - u_{M,N} \right\|_{L^\infty(\Omega)}.$$

Example 1. We take the test function

$$U(x,\theta) = (1-x)(1+x)^2 e^{x+\sin\theta} + (x^2+2x-3)\cos\theta + 1,$$

and $\mu = 1$. In Fig. 1, we plot the numerical errors $\log_{10} E_{M,N,1}$ and $\log_{10} E_{M,N,2}$ vs M with $N = 2M$, respectively. They demonstrate that the numerical errors decay exponentially as $N \to \infty$. This fact coincides well with the theoretical analysis.

Example 2. We take the test function

$$U(x,\theta) = (1-x)(1+x)^2 \sin(x+\theta) + 1,$$

and $\mu = 1$. In Fig. 2, we plot the numerical errors $\log_{10} E_{M,N,1}$ and $\log_{10} E_{M,N,2}$ vs M with $N = 2M$, respectively. They also show that the numerical errors decay exponentially as $N \to \infty$.

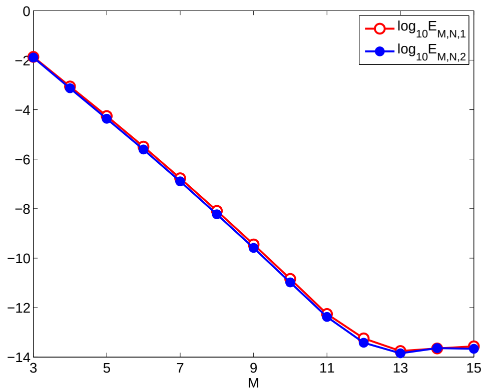

Figure 1: The discrete L^2- and $L^\infty-$errors.

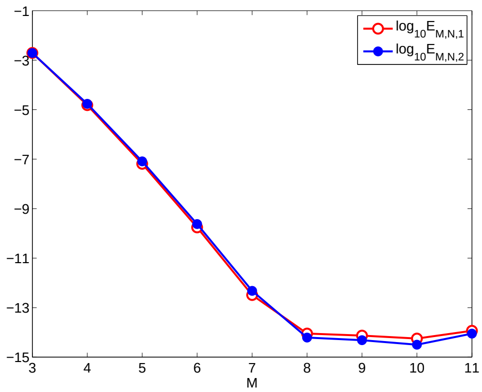

Figure 2: The discrete L^2- and $L^\infty-$errors.

6. Concluding Remarks

In this paper, we proposed a Fourier-Jacobi spectral method for two-dimensional Neumann problems. The mixed Fourier-Jacobi orthogonal approximation was established. The numerical error of the proposed spectral scheme was analyzed. In particular, by choosing appropriate base functions with zero slope at the boundary, the stiffness matrix is tridiagonal, rather than a full matrix by using the classical spectral method. The numerical results demonstrated the spectral accuracy of proposed schemes, and coincided well with the theoretical analysis.

Acknowledgments

This work is supported in part by Shuguang Project of Shanghai Education Commission, N.08SG45, and The Fund for E-institute of Shanghai Universities N.E03004.

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