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Fluxon Centering in Josephson Junctions with Exponentially Varying Width

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Abstract. Nonlinear eigenvalue problems for fluxons in long Josephson junctions with exponentially varying width are treated. Appropriate algorithms are created and realized numerically. The results obtained concern the stability of the fluxons, the centering both magnetic field and current for the magnetic flux quanta in the Josephson junction as well as the ascertaining of the impact of the geometric and physical parameters on these quantities. Each static solution of the nonlinear boundary-value problem is identified as stable or unstable in dependence on the eigenvalues of associated Sturm-Liouville problem. The above compound problem is linearized and solved by using of the reliable Continuous analogue of Newton method.

AMS subject classifications: 34L16, 34K10, 65D30, 65N12, 65N25, 65F15

Key words: Fluxon, stability, bifurcation, critical curve, centering magnetic field, centering current.

1. Introduction

In 1962 Josephson observed a supercurrent (i.e. a current that may flow for an indefinitely long time without any applied voltage) across a device subsequently called a Josephson junction (JJ), consisting of two superconductors coupled by a weak link. This discovery, now known as the Josephson effect, led to many numerical and empirical investigations of one-stacked or multi-stacked homogeneous and inhomogeneous JJs [2, 9–11, 16]. There are several ways to model the shape and the influence of the inhomogeneity — e.g. by a Dirac delta function, some finite piece-wise function, or some smooth function. The JJs with variable geometry, typically modelled by smooth functions, have recently received considerable attention [3, 5, 8, 13] and are the subject of this article.

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A basic feature of any JJ is the critical (bifurcation) dependence of the "current-magnetic field", directly related to the stability of the magnetic flux quanta. The respective boundary and eigenvalue problems are quite difficult to solve analytically, even in the one-dimensional case, and this has led to the development and implementation of appropriate numerical procedures. Actually, the stable solutions that are sought that describe so-called fluxons turn out to be admissible space-time distributions in the JJ. However, there are difficulties due to the presence of physical and geometrical parameters, which can significantly affect the solution behaviour. Thus when critical modes of the distributions are considered, the corresponding parameter set should be regarded as a parametric eigenspace that complicates the investigation

Our interest in these modes has also been fostered by the empirical investigations conducted by Benabdalah *et al.* [3]. They found a threshold for the current, beyond which the static distribution becomes unstable with fluxons moving to the narrower end of the device. Their observations lead us to conjecture that the outer magnetic field and current influence and control the location and stability of the Josephson vortices in the JJ. With this in mind, we consider the fluxons in inhomogeneous JJs with exponentially varying width — in particular, the stability of the quant-magnetic field, the centering of the magnetic field and the current, and the relationships and dependencies between them and quantities in the multiparametric space. In Section 2, we pose boundary-value problems for in-line and overlap geometries, and appropriate eigenvalue problems. The linearisation, discretisation and the numerical procedure adopted are considered in Section 3. Our main results, their interpretation, and a final discussion are then presented Sections 4 and 5.

2. Formulation of the Problem

A sine-Gordon wave equation is assumed to govern the space-time evolution of the magnetic flux in the JJ with exponentially varying width [3]. For static distributions of the magnetic flux in JJs with in-line and overlap geometries, we consider two different two-point boundary-value problems (BVP) — viz.

$$\begin{aligned} &-\varphi_{xx}+\sigma\varphi_x+\sin\varphi-\sigma h_e=0\,,\\ &\varphi_x(0)-h_e+l\gamma=0\,,\qquad \varphi_x(l)-h_e=0\,; \end{aligned} \tag{2.1}$$

$$\begin{aligned} &-\varphi_{xx}+\sigma\varphi_x+\sin\varphi-\sigma h_e-\gamma=0\,,\\ &\varphi_x(0)-h_e=0\,,\qquad \varphi_x(l)-h_e=0\,. \end{aligned} \tag{2.2}$$

Here l is the the length of the JJ; and φ_x denoting the normed magnetic field in the junction, h_e the intensity of the outer (bias) magnetic field, γ the density of the normed outer current, and σ the shape parameter, are all dimensionless quantities. [We suppose the width of the JJ varies according the law $W(x)=W_0\exp(-\sigma x)$.]

In solving the BVP (2.1) or (2.2), we need information about the stability of the sought solution. Moreover, we are interested in the minimal stable state, especially the critical "current-magnetic field" relationship [13]. Given the nonlinearity (2.1) and (2.2) , we might not expect to obtain a unique solution. Indeed, numerous numerical implementations show that the BVP admit more than one solution $\varphi=\varphi(x,p)$ for a given parametric set $p=\{l,\sigma,\gamma,h_e\}$, where each solution corresponds to a stable spatial distribution of the magnetic flux. The stability depends upon the actual values of the parametric set p, and the critical values of the current and the outer magnetic field determine the limits of the stability for given h_e and γ. From the form of the partial differential equation in the BVP (2.1) or (2.2), in order to ascertain the stability of a given static solution we consider the corresponding eigenvalue problem

$$-\psi_{xx}+\sigma\psi_x+q(x)\psi=\lambda\psi,\quad \psi_x(0)=0,\quad \psi_x(l)=0. \tag{2.3}$$

Since $q(x)\equiv\cos\varphi_s(x)$ is a bounded function and the interval $[0,l]$ is finite, this is a regular Sturm-Liouville problem with a discrete spectrum of eigenvalues bounded from below (e.g., see [15]). In order to get a unique eigenfunction $\psi_n(x)$ corresponding to a given eigenvalue λ_n $(n=1,2,3,\ldots)$, we invoke the norm condition

$$\int_0^l \psi_n(x)\mathrm{d}x=1. \tag{2.4}$$

Since the solution of the BVP (2.1) or (2.2) evidently depends upon the parameters in p, we anticipate that the eigenquantities also depend upon these parameters — i.e. we anticipate eigenvalues $\lambda_n(p)$ and eigenvectors $\psi_n(x,p)$. The criterion for stability of a given static solution is the sign of the minimal eigenvalue — i.e. if $\lambda_{\min}>0$ then the corresponding solution $\varphi(x,p)$ is stable, and if $\lambda_{\min}<0$ it is unstable. With this in mind, we call the current (driven by a magnetic flux) of magnitude γ critical when $\lambda_{\min}(p)\equiv 0$. When the current reaches this critical value, the fluxon in JJ is destroyed and a localised state no longer exists. This corresponds to a transition (bifurcation) from a stable to an unstable state (solution). The graph of the implicit relationship $\lambda_{\min}(\gamma,h_e)=0$ describes the critical (bifurcation) curve of the "current-magnetic field."

We denote the stable solutions by M for the Meissner's solution, Φ^1 for the main fluxon, and Φ^n, $n=2,3,4,\cdots$ for the multifluxon solutions, respectively. The digit n determining the number of fluxons (NoF) depends upon the value of the functional

$$N[\varphi]=\frac{1}{\pi l}\int_0^l \varphi(x,p)\mathrm{d}x. \tag{2.5}$$

Our numerical calculations show that the value of $N[\varphi]$ is approximately one for the main fluxon, two for the two-fluxon state, etc. — cf. Fig. 1, where the stable solutions M, Φ^1, Φ^2, etc. are plotted by solid lines and the unstable solutions by dashed lines. Some magnitude h_m of the outer magnetic field h_e corresponds to each fluxon (all the other parameters in the set p are fixed). Thus the relevant value of $N[\varphi]$ seems to be some digit, and the maximum of the derivative $\varphi_x(x)$ is then localised at the center $x=l/2$ of JJ. We call the quantity h_m a *centering magnetic field*. The geometric sense of this centering field is readily seen in Fig. 2, where the graphs of the magnetic field $\varphi_x(x)$ of the main fluxon are shown for three different values of h_e.

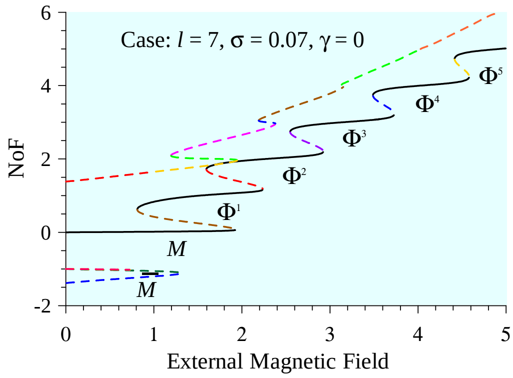

Figure 1: Functional $N[\varphi]$ (fluxon number NoF).

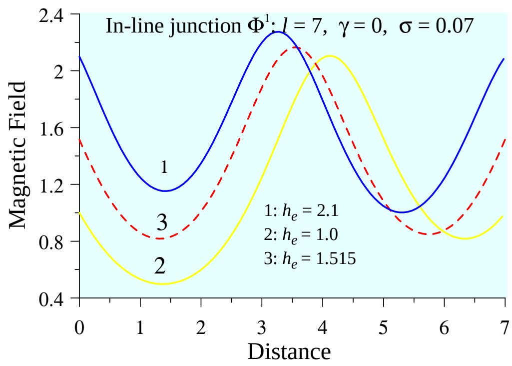

Figure 2: Centering field $h_m = 1.515$ (curve 3).

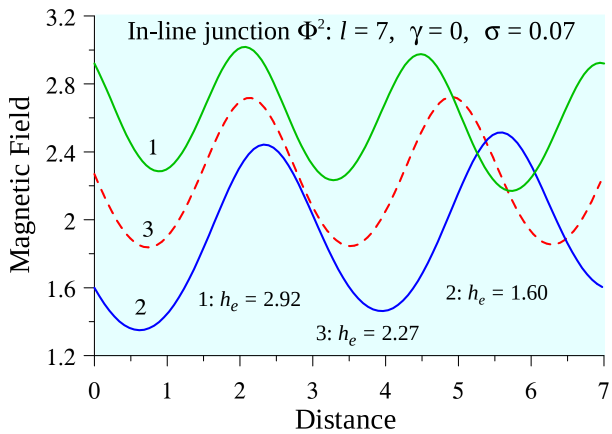

Figure 3: Influence of the magnetic field h_e over Φ^2.

It is evident that the centering is possible for large enough magnitudes of h_e, which push out the fluxon Φ^2 to the left but keep its shape. For small magnitudes of the magnetic field the fluxon is localised to the right end of the JJ. The influence of h_e over the two-fluxon spatial distribution of the magnetic field is the same (cf. Fig. 3), where the centering magnetic fields h_m are of magnitude h_e, tending to make the graph of the derivative $\varphi_x(x)$ symmetric about the center of the JJ with $N[\Phi^2] = 2$ Josephson vortices. Similarly, we can define and generalise the quantity "centering current" γ_m as the magnitude of the current γ that "digitises" the functional N for other fixed parameters.

Based on these considerations, it appears that the magnetic field can be sought as a solution of the nonlinear BVP (2.1) or (2.2). Thus for a given set of current γ and geometric parameters l and σ, the solution satisfies Eq. (2.5) for some digit n. To explore the critical modes, information on the stability of the solution can be obtained from the corresponding uniform Sturm-Liouville eigenvalue problem (EVP) (2.3). Since we are interested in the unknown function $\varphi(x)$ and the unknown quantity h_e, the combined BVP and EVP can be interpreted as a generalised EVP — i.e. we can pose an EVP with respect to the centering current γ_m. In all cases, we need a norm condition such as Eq. (2.4) to determine the eigenfunctions $\psi_n(x)$ uniquely. Similarly, we can define the centering current γ_m.

3. Linearization and Algorithmic Implementation

To identify the centering magnetic field (centering current), we implement the Continuous Analogue of the Newton Method (CANM) for a vector functional equation (e.g. see [12] or [17]). In particular, for a JJ with an overlap geometry, Eqs. (2.2) and (2.5) can be rewritten as the functional equation $\vec{f}(\varphi, h_m) = 0$ involving the vector-function

$$\vec{f}(\varphi, h_m) = \begin{cases} -\varphi_{xx} + \sigma(\varphi_x - h_e) + \sin\varphi - \gamma \\ \varphi_x(0) - h_m \\ \varphi_x(l) - h_m \\ \frac{1}{\pi l}\int_0^l \varphi(x,p)\mathrm{d}x - n \end{cases} . \tag{3.1}$$

We suppose the pair (φ^*, h_e^*) is an isolated solution that is a limit of the convergent functional sequence $\{\varphi^n, h_e^n\}$ — i.e. $\{\varphi^n, h_e^n\} \to (\varphi^*, h_e^*)$. Further, we can parameterise the pairs as $\{\varphi^n, h_e^n\} = \{\varphi^n(\theta), h_e^n(\theta)\}$, where the real θ varies continuously. Thus we obtain the so-called Gavurin equation, which is a quantitative implementation of the CANM:

$$\vec{f}'_{\varphi}(\varphi, h_m)\,\Phi + \vec{f}'_{h_m}(\varphi, h_m)\,H + \vec{f}(\varphi, h_m) = 0, \quad \dot{\varphi} - \Phi = 0, \quad \dot{h}_m - H = 0\,.$$

Here $(\cdot)'$ denotes the Fréchet derivative and $(\dot{\cdot})$ differentiation with respect to θ. The direct solution of the Gavurin equation is a hard task, so we linearise — cf. our previous work (e.g. [6]) for details. Thus we get two linear two-point BVPs

$$\begin{aligned} &-u_{xx} + \sigma u_x + \cos\varphi\, u = \varphi_{xx} + \sigma\left(h_m - \varphi_x\right) - \sin\varphi + \gamma, \\ &u_x(0) = h_m, \qquad u_x(l) = h_m\,; \end{aligned} \tag{3.2}$$

$$\begin{aligned} &-v_{xx} + \sigma v_x + \cos\varphi\, v = 1, \\ &u_x(0) = 1, \qquad u_x(l) = 1\,, \end{aligned} \tag{3.3}$$

with the same left-hand sides. The unknown functions $u = u(x,\theta)$ and $v = v(x,\theta)$ are components of the decomposition $\Phi = u + Hv$. When (3.2) and (3.3) have been solved, we find the derivative $H(\theta)$ from the explicit formula

$$H = \left(\int_0^l v(x)\,\mathrm{d}x\right)^{-1}\left[\pi l n - \int_0^l \varphi(x)\,\mathrm{d}x - \int_0^l u(x)\,\mathrm{d}x\right],$$

which follows from the norm condition Eq. (2.5) after applying the CANM.

The iterative procedure involved is considered in Refs. [6] and [7], so we will not repeat its further description and explanation here. However, we recall that the discretisation is based on the Euler method [1], and that the collocation involves Hermitian splines [4].

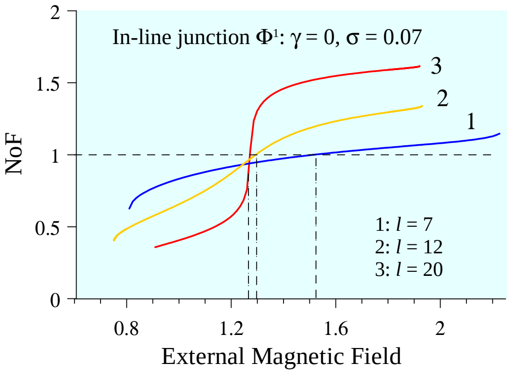

Figure 4: Functional $N(l)$ vs. external magnetic field h_e (Heaviside step function for $h_e = h_m$).

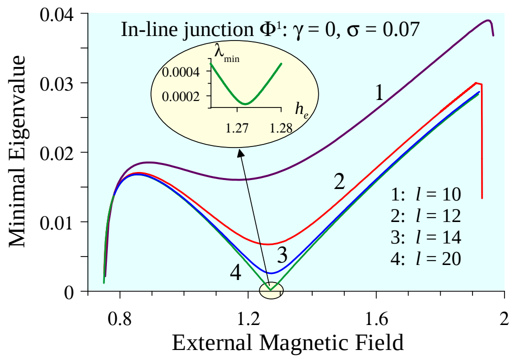

Figure 5: The minimal eigenvalue $\lambda_{\min}(h_e)$, for fixed lengths l of the JJ.

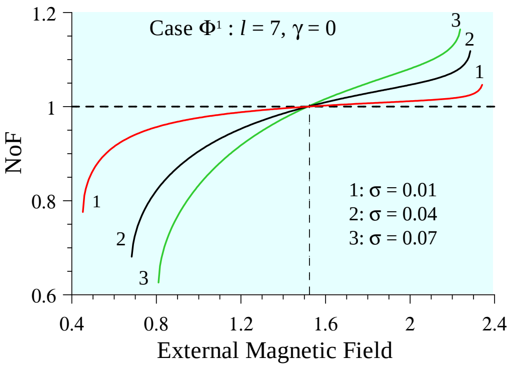

Figure 6: The functional $N(h_e)$ vs. the shape parameter σ.

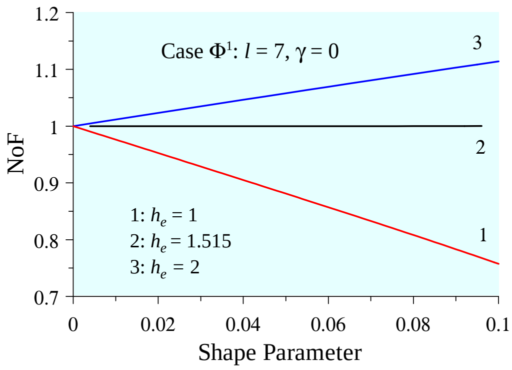

Figure 7: The functional $N(\sigma)$ vs. the external magnetic field h_e.

4. Results and Discussion

Based on the linearisation and consequent numerical algorithm as developed in [17], we calculate the value of the centering magnetic field for various values of the parameters in the model. When $\sigma = 0.07$, $l = 7$ and $\gamma = 0$ the computed centering magnetic field $h_m \approx 1.515$ for the main fluxon Φ^1, while the two-fluxon bound state Φ^2 has a centering magnetic field $h_m \approx 2.2689$.

In particular, we examined the influence of the geometric parameters (the length l and shape parameter σ) on h_m and γ_m. If the length of the JJ increases to $l = 12$ and both σ and the current are unchanged, the magnetic field corresponding to the main fluxon is $h_m \approx 1.293$. However, as the length l increases h_m decreases, such that when $l = 20$ the computed magnitude of the centering magnetic field is $h_m \approx 1.273$. The results illustrated Fig. 4 clearly show how the function $N(l)$ tends towards a Heaviside step function centered

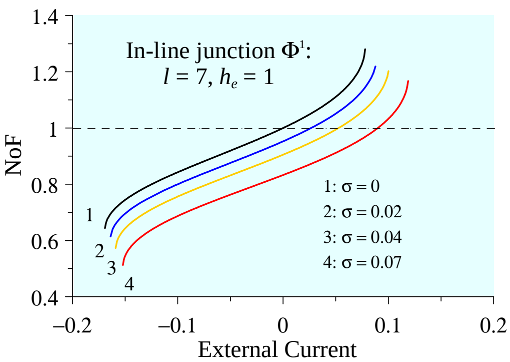

Figure 8: The functional $N(\gamma)$ vs. the shape parameter σ – in-line geometry.

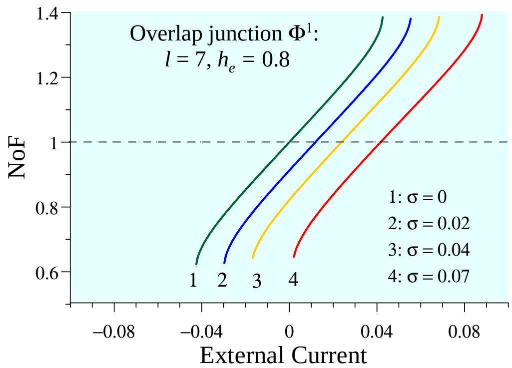

Figure 9: The functional $N(\gamma)$ vs. the shape parameter σ – overlap geometry.

at the point h_m for larger values of the length l, reflecting the relationship between the minimal eigenvalue and the outer field h_e. When the length l of the JJ increases, the graph of the function $\lambda_{\min}(h_e)$ tends to approach the horizontal axis h_e such that the function possesses more than two zeros for large enough lengths ($l \gg 1$) as illustrated in Fig. 5.

The influence of the shape parameter σ on the function $N(h_e)$ and the centering magnetic field h_e was then investigated, and some results are shown in Figs. 6 and 7. Each graph in Fig. 6 corresponds to a different interval of variation of the field h_e with a fixed shape parameter σ where the main fluxon is stable. For instance, when $\sigma = 0.07$ the respective interval of stability of the external magnetic field is $h_e \in (0.81; 2.238)$. We observe that the intervals of stability of the main fluxon become narrower when σ increases, regardless of the intensity of the centering magnetic field. The curves have a common point of intersection, corresponding to the centering magnetic field keeping the fluxon localised in the middle of the JJ. This directly reflects the behaviour of the function $N(\sigma)$ shown in Fig. 7 (cf. Line 2). Two more graphs are plotted there for different values of the external field h_e. When $h_e > h_m$ (Line 3), the dependence of $N(\sigma)$ is linear with positive slope. From the physical point of view, this may be interpreted as a shift of the magnetic field quanta to the left end of the JJ as the shape parameter σ increases. The linear fall (negative slope) of the function $N(\sigma)$ for $h_e < h_m$ is also evident, when the fluxon then moves to the other end of the JJ.

We also investigated the influence of the shape of the JJ on the magnitude of the centering current γ_m when the length l and the external magnetic field h_e are fixed. Figs. 8 and 9 describe both geometries, in-line and overlap, for a comparative study. A typical feature of the JJ with exponentially varying width is the availability of a “geometric current” $g(x) = \sigma[\varphi_x(x) - h_e]$ [13] that pushes out the fluxons to the end of the JJ. For magnetic fields with small magnitude, the fluxons are drawn to the left; and to be centered, they need a positive centering current γ_m to compensate and nullify the influence of the geometric current. An increasing shape parameter σ induces an increased geometric current $g(x)$, as well as an increased centering current γ_m. The JJ with in-line geometry appears

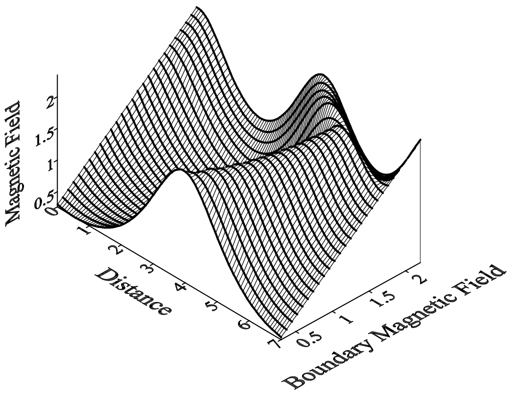

Figure 10: Bifurcation surface "current-magnetic field" for the JJ with overlap geometry.

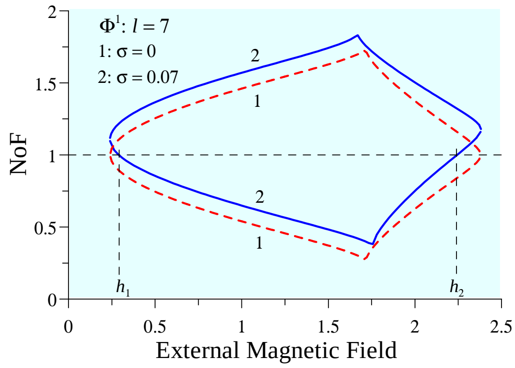

Figure 11: Functional N (NoF) vs. the external magnetic field h_e.

to be more sensitive, however.

In order to construct the critical (bifurcation) dependent "current-magnetic field", one must first investigate the bifurcation distributions of the magnetic field. Fig. 10 shows the computed bifurcation solutions for the JJ with an overlap geometry and length $l = 7$, as functions of the critical h_e and γ, and illustrates the location and the kind of the fluxon before their destruction by the injection current.

Finally, we considered the features of the relationship $N(h_e)$ to the bifurcation distributions of the magnetic field, for both positive and negative values of the current γ (cf. Fig. 11). In contrast to the case $\sigma = 0$, for $\sigma = 0.007$ the extreme (minimal and maximal) points of the magnetic field h_e do not lie on the horizontal line $N = 1$. There are now intervals in which the outer magnetic field varies, where one not only needs to inject an extra current γ in order to balance the "geometric" current but also to insert into the JJ a quantum of magnetic flux. This is confirmed by the location of the bifurcation curves modelled numerically, and as investigated in [5, 14].

5. Concluding Remarks

We have primarily explored stable static distributions of the magnetic field in Josephson junctions with exponentially varying width, for both in-line and overlap geometries. We have investigated the stability of the magnetic field by complementing the appropriate nonlinear boundary-value problem with a conjugate Sturm-Liouville problem, and then treating the resulting composite problem as an united functional equation. By using of the Continuous Analogue of Newton Method, we have built an efficient algorithm to solve this numerically. Our results cover the influence of the geometric parameters length l and shape σ on the centering magnetic field h_m and centering current γ_m. An increasing shape parameter σ narrows the interval of stability of the main fluxon, but does not simultaneously affect the magnitude of the centering magnetic field. However, both the geometric current and the centering current γ_m increase. The dependence is more sensitive for the JJ with in-line geometry. The outer magnetic field affects the stability of the main fluxon supported by an injection current, which compensates for the potentially destabilising influence of the geometry on the fluxon.

The algorithms developed for identifying a centering current and a magnetic field represent a good foundation for constructing JJ models with more inhomogeneities and more general geometry, and may provide a suitable tool for further investigation of the critical modes and transitions between states in Josephson devices.

Acknowledgments

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