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On Solution Regularity of Linear Hyperbolic Stochastic PDE Using the Method of Characteristics

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Abstract. The generalized Polynomial Chaos (gPC) method is one of the most widely used numerical methods for solving stochastic differential equations. Recently, attempts have been made to extend the the gPC to solve hyperbolic stochastic partial differential equations (SPDE). The convergence rate of the gPC depends on the regularity of the solution. It is shown that the characteristics technique can be used to derive general conditions for regularity of linear hyperbolic PDE, in a detailed case study of a linear wave equation with a random variable coefficient and random initial and boundary data.

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1. Introduction

Stochastic differential equations are of great importance, not only in a wide variety of scientific areas including non-equilibrium statistical physics and quantitative biology but also elsewhere such as in mathematical finance and economics. Since it is usually difficult to solve stochastic partial differential equations (SPDE) analytically, their numerical solution has generated considerable research interest in recent decades. A good introduction and review can be found in Ref. [2].

Among a number of proposed numerical methods, the generalized Polynomial Chaos (gPC) method is one of the most widely used. The basic idea of the gPC is to project the random part of the system into a finite dimensional polynomial space and then solve the projected finite dimensional deterministic system. Due to its spectral-method-like nature, one of the important properties of the gPC is that the rate of the convergence depends very much on the regularity of the solution. An excellent review of the gPC method can be found in Ref. [6]

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The gPC method has proven to be very efficient for various types of problems, but only recently have attempts been made to use it for hyperbolic SPDE [1,3,5]. While parabolic and elliptic equations generally have good regularity properties, hyperbolic problems have non-degenerate characteristics that can depend on the randomness, making the study of their regularity more difficult. This paper develops the characteristics technique to deal with the regularity of the solution to a particular linear SPDE, as a first step to tackle the regularity of hyperbolic SPDE in general.

The strategy of the characteristics technique is outlined in Section 2. In Section 3, the technique is used to derive a general condition for the BV regularity of the solution to the linear wave equation with a random variable coefficient and random initial and boundary data. Conclusions are made in Section 4.

2. Outline of the Characteristic Approach

An energy-estimate-type technique can be applied to linear hyperbolic SPDE to show BV regularity [5] . However, this technique requires some strong conditions. In the case of a wave equation with a constant random wave speed, the derivative of the wave speed $c_y(y)$ must be bounded. Since the method of characteristics generally gives sharp estimates for linear hyperbolic equations, we intend to use it to improve the aforementioned result and obtain a more general theorem.

The idea of the characteristics approach is straightforward. Thus we find an implicit representation of the solution in terms of the given data using the characteristics and then bound its TV norm. For a general hyperbolic PDE with a random parameter but no shock, the characteristics can be written as

$$L(x,t,y) = s$$

and initial data can be parametrized by a random curve

$$\Gamma : \gamma(s) = (f(s), g(s), h(s,y)) = (x,t,u(y)) ,$$

where y is a random variable. The first equation contains the information about the influence of the randomness on the directions of the characteristic curves, and the second equation contains the randomness of the initial data. Then the solution can be written as

$$u(x,t,y) = h(L(x,t,y),y) ,$$

and we see immediately that

$$\left\|u_y\right\|_{L^1} = \int_{-1}^{1} \rho(y) \int_{-1}^{1} \left| \frac{\partial h}{\partial s}\frac{\partial L}{\partial y} + \frac{\partial h}{\partial y} \right| dx\, dy ,$$

where the physical interpretation of each term is self-evident.

Although the idea of the technique is simple, as shown below it is nontrivial and needs careful analysis, even when it is applied to a relatively simple problem. The major obstacle is the interaction between the boundary and the characteristics, which makes the parametrization of the solution in terms of initial data difficult.

3. Regularity of Wave Equation with a Variable Random Wave Speed

We consider the regularity of the following SPDE problem:

$$\begin{cases} u_t(x,t,y) - c(x,y)u_x(x,t,y) = 0\,, & \forall (x,t,y) \in (-1,1)\times(0,T]\times[-1,1]\,, \\ u(-1,t,y) = u_L(t,y)\,, & \text{for } c(x,y) < 0 \text{ and } t \in (0,T]\,, \\ u(1,t,y) = u_R(t,y)\,, & \text{for } c(x,y) > 0 \text{ and } t \in (0,T]\,, \\ u(x,0,y) = u_0(x,y)\,, & \forall (x,y) \in [-1,1]\times[-1,1]\,, \end{cases} \tag{3.1}$$

where $y \in [-1,1]$ is a random variable with a distribution given by its probability density function $\rho(y)$. It is assumed that the wave speed $c(x,y)$ does not change sign when x varies — i.e. for almost all fixed $y \in [-1,1]$, either $c(x,y) > 0$ or $c(x,y) < 0, \forall x \in (-1,1)$. It will be shown that this problem is well-posed.

We seek a condition such that the solution $u(x,t,y)$ is in $W^{1,1} \in BV$ with respect to y — i.e.

$$\left\|u_y\right\|_{L^1} = \int_{-1}^{1} \rho(y) \int_{-1}^{1} \left|u_y(x,t,y)\right| dx\, dy < \infty\,,$$

so (3.1) may be solved efficiently using the gPC method with collection projection [5,6]. The case where $c(x,y) = c(y)$ is independent of x is studied in Ref. [5], where necessary conditions for the solution to be in H^n and BV are given. Since the general problem to be considered eventually will be nonlinear, the more natural BV space and BV regularity are considered here, but the techniques discussed can easily be extended to H^n regularity.

In the rest of this Section, the well-posedness of the problem is considered and proof of the existence and uniqueness of the solution is given, followed by proof of the associated BV condiion. The proof is somewhat technical, so the ultimate main result is stated first.

3.1. Main result

Theorem 3.1. *If*

1.

$$c_M(y) \overset{def}{=} \sup_{x\in(-1,1)} \left|c(x,y)\right| < \infty \quad \textit{for almost all} \;\; y \in [-1,1]\,;$$

2.

$$d(y) \overset{def}{=} \int_{-1}^{1} \left| \frac{\partial}{\partial y}\left(\frac{1}{c(\xi,y)}\right)\right| d\xi = \int_{-1}^{1} \left|\frac{c_y(\xi,y)}{c^2(\xi,y)}\right| d\xi < \infty \quad \textit{for almost all} \;\; y \in [-1,1]\,;$$

and

3.

$$\int_{c(-1,y)<0} \rho(y)c_M(y) \int_0^T d(y) \left|\frac{\partial u_L}{\partial t}(\xi,y)\right| + \left|\frac{\partial u_L}{\partial y}(\xi,y)\right| d\xi dy < \infty \,,$$

$$\int_{c(1,y)>0} \rho(y)c_M(y) \int_0^T d(y) \left|\frac{\partial u_R}{\partial t}(\xi,y)\right| + \left|\frac{\partial u_R}{\partial y}(\xi,y)\right| d\xi dy < \infty \,,$$

$$\int_{-1}^{1} \rho(y)c_M(y) \int_{-1}^{1} d(y) \left|\frac{\partial u_0}{\partial x}(\xi,y)\right| + \left|\frac{1}{c(\xi,y)}\frac{\partial u_0}{\partial y}(\xi,y)\right| d\xi dy < \infty \,,$$

then the solution $u(x,t,y)$ to the SPDE (3.1) *is in $W^{1,1} \in BV$:*

$$\left\|u_y\right\|_{L^1} < \infty \,.$$

The first condition means that $c(x,y)$ is bounded, the second condition that the influence of the change of the characteristics in x can be bounded, and the third give conditions on the regularity of the initial data.

The proof of this main result is given in Subsection 3.3. However, when $c(x,y) = c(y)$ there is an interesting result.

Corollary 3.1. *The solution $u(x,t,y)$ to the SPDE* (3.1) *given $c(x,y) = c(y)$ is in BV, if the following conditions are satisfied:*

$$\int_{c(y)<0} \rho(y)\frac{c'(y)}{c(y)} \int_0^T \left|\frac{\partial u_L}{\partial t}(\xi,y)\right| d\xi dy < \infty \,,$$

$$\int_{c(y)<0} \rho(y)c(y) \int_0^T \left|\frac{\partial u_L}{\partial y}(\xi,y)\right| d\xi dy < \infty \,,$$

$$\int_{c(y)>0} \rho(y)\frac{c'(y)}{c(y)} \int_0^T \left|\frac{\partial u_R}{\partial t}(\xi,y)\right| d\xi dy < \infty \,,$$

$$\int_{c(y)>0} \rho(y)c(y) \int_0^T \left|\frac{\partial u_R}{\partial y}(\xi,y)\right| d\xi dy < \infty \,,$$

$$\int_{-1}^{1} \rho(y)c(y) \int_{-1}^{1} \left|\frac{\partial u_0}{\partial x}(\xi,y)\right| d\xi dy < \infty \,,$$

$$\int_{-1}^{1} \rho(y) \int_{-1}^{1} \left|\frac{\partial u_0}{\partial y}(\xi,y)\right| d\xi dy < \infty \,.$$

Proof. The proof is immediate, with $c_M(y) = c(y)$ and $d(y) = 2\left|c'(y)/c^2(y)\right|$. □

This corollary is stronger than than the corresponding result derived in Ref. [5], since the condition that $c'(y)$ be bounded is removed.

Corollary 3.2. *If $c_y(x,y)$ and $c(x,y)$ are bounded almost everywhere, when $c(x,y)=c(y)$ the solution $u(x,t,y)$ to the SPDE* (3.1) *is in BV if the following conditions are satisfied:*

$$\int_{c(-1,y)<0}\rho(y)\int_0^T\left|\frac{1}{c(-1,y)}\right|\left|\frac{\partial u_L}{\partial t}(\xi,y)\right|+\left|\frac{\partial u_L}{\partial y}(\xi,y)\right|d\xi dy<\infty\,,$$
$$\int_{c(1,y)>0}\rho(y)\int_0^T\left|\frac{1}{c(1,y)}\right|\left|\frac{\partial u_R}{\partial t}(\xi,y)\right|+\left|\frac{\partial u_R}{\partial y}(\xi,y)\right|d\xi dy<\infty\,,$$
$$\int_{-1}^{1}\rho(y)\int_{-1}^{1}\left|\frac{\partial u_0}{\partial x}(\xi,y)\right|+\left|\frac{1}{c(\xi,y)}\frac{\partial u_0}{\partial y}(\xi,y)\right|d\xi dy<\infty\,.$$

Proof. Since $c_M \le C$ is bounded almost everywhere, the constant factor may be dropped; and since $c_y \le C$ is also bounded almost everywhere,

$$\begin{aligned} d(y) &= \int_{-1}^{1}\left|\frac{c_y(\xi,y)}{c^2(\xi,y)}\right|d\xi<\infty \\ &\le C\int_{-1}^{1}\left|\frac{1}{c^2(\xi,y)}\right|d\xi<\infty \\ &= C\left|\frac{1}{c(-1,y)}-\frac{1}{c(1,y)}\right|\,. \end{aligned}$$

The $\rho(y)dy$ integral is restricted to either $c>0$ or $c<0$ so that only one of the two terms enters the final integral. Then on dropping the constant factors, we obtain the result. □

This result can also be derived using energy arguments [5], if c_x is also bounded almost everywhere.

In brief, the characteristics approach provides a sharper estimate and leads to a more general theorem.

3.2. Well-posedness of the problem and the solution representation

The existence and uniqueness of the solution to our SPDE (3.1) follows from a solution representation in terms of the initial data and the characteristics.

3.2.1. Parametrization of the initial data

We parametrize the initial data using a curve Γ — viz.

$$\Gamma:\gamma(s,y)=(f(s),g(s),h(s,y))\,,$$

where

$$f(s) = \begin{cases} -1\,, & s < -1\,, \\ s\,, & s \in [-1,1]\,, \\ 1\,, & s > 1\,; \end{cases}$$

$$g(s) = \begin{cases} -(s+1)\,, & s < -1\,, \\ 0\,, & s \in [-1,1]\,, \\ s-1\,, & s > 1\,; \end{cases} \tag{3.2}$$

$$h(s) = \begin{cases} u_L(-s-1,y)\,, & s < -1 \text{ and } c(-1,y) < 0\,, \\ u_0(s,y)\,, & s \in [-1,1]\,, \\ u_R(s-1,y)\,, & s > 1 \text{ and } c(1,y) > 0\,. \end{cases}$$

This curve Γ is the boundary of the strip of interest $(x,t) \in (-1,1) \times (0,\infty)$.

3.2.2. Characteristics

It is convenient to index the characteristics of the SPDE using $(x_0(s), t_0(s))$ instead of s alone, as will become clear later. The characteristics satisfy

$$\frac{dt}{dx} = -\frac{1}{c(x,y)}\,, \qquad x(t_0) = x_0\,, \tag{3.3}$$

and therefore are given by

$$K(x,t,s,y) = J(x,t,x_0(s),t_0(s),y) = (t - t_0) + \int_{x_0}^{x} \frac{d\xi}{c(\xi,y)} = 0\,,$$

where $(x_0(s), t_0(s))$ is on Γ.

Lemma 3.1. *The characteristics given by $K(x,t,s,y) = 0$ cannot cross each other, so there is no shock. Further, each characteristic crosses the initial curve Γ only once, and the point of intersection $(x_0(s), t_0(s))$ is unique.*

Proof. From the elementary theory of ODE, two integral curves from (3.3) with two different starting points cannot cross unless the two points are on the same integral curve. Thus the characteristics are well behaved, unlike curve 1 in Fig. 1. Indeed, from the expression for $K(x,t,s,y) = 0$ we have

$$t(x;t_0,x_0,y) = t_0 - \int_{x_0}^{x} \frac{d\xi}{c(\xi,y)}\,,$$

which is clearly a single-valued function in x.

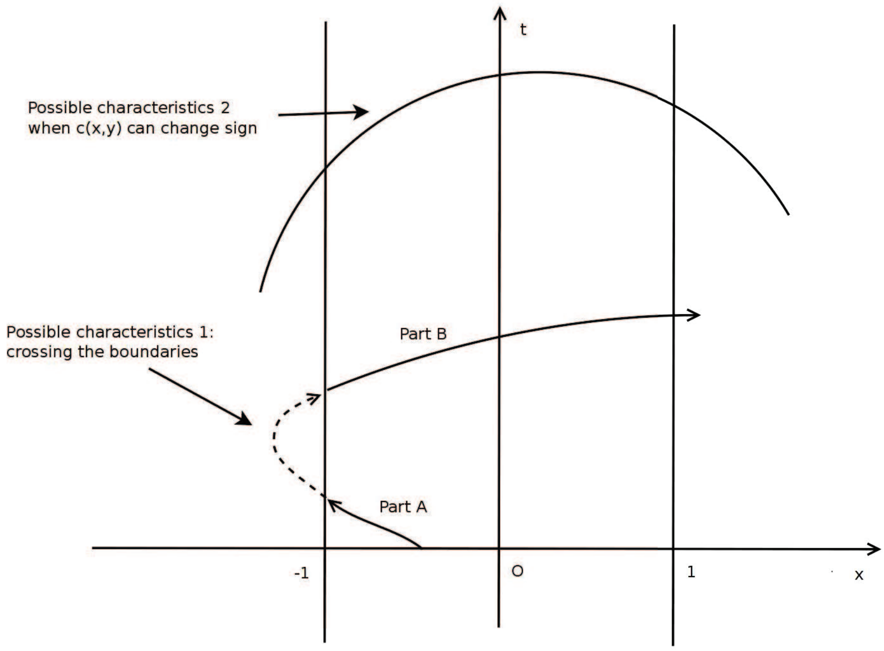

Figure 1: Illustration of possible bad characteristics.

Let us now prove the second part of the lemma. Since $c(x,y)$ is positive or negative $\forall x \in (-1,1)$ for a fixed y, the function $t = t(x;t_0,x_0,y)$ for the characteristics is monotonic. When $c(x,y) < 0$, we only have the boundary at $x = -1$ and $t(x;t_0,x_0,y)$ is increasing. Now any characteristic curve starting from the x-axis cannot cross $x = -1$, because it is single-valued and cannot cross itself (monotonicity). Similarly, any characteristic curve stemming from $x = -1$ cannot cross the x-axis nor itself because $t = t(x;t_0,x_0,y)$ is single-valued. The case for $c(x,y) > 0$ is similar. Thus the characteristics cross the initial curve Γ only once, at the starting point. $\square$

Remark 3.1. It is notable that the wave speed $c(x,y)$ cannot change sign as x varies, for otherwise there may be physical and mathematical deficiencies. For example, if $c(x,y) = y/x$ there is a sign change through ∞, so there are ambiguous characteristics like "Possible characteristics 2" in Fig. 1 where the wave propagate from the left to the right or in the opposite direction. As another example, if $c(x,y)$ changes sign through 0 like $c = x - x_0$, then the solution has two independent parts separated by the straight line $x = x_0$. In general, $x_0 = x_0(y)$ also depends on the random variable y, so it is much harder to bound $\|u_y\|_{TV}$.

3.2.3. Existence, uniqueness and representation of the solution

Let us first prove there is no rarefaction so that every point in the strip of interest $(x,t) \in (-1,1) \times (0,T]$ can be reached by a characteristic, and then from Lemma 3.1 that the inverse of the characteristics function is well-defined. This leads us to the representation of the solution in terms of the initial data and the characteristics.

Lemma 3.2. *There is no rarefaction — i.e.*

$$\forall (x_p, t_p) \in (-1,1)\times(0,T]\,, \quad \exists s_p \in [-1-t_p, 1+t_p] \text{ such that } K(x_p,t_p,s_p,y) = 0\,,$$

so the value of the solution can be found in the initial data through the characteristics at every point. (Note that $s_p \in [-1-t_p, 1+t_p]$ means the solution also satisfies the causality condition.)

Proof. First consider $c(x,y) < 0$, when

$$\int_{-1}^{x_p} \frac{d\xi}{c(\xi,y)} \le \int_{x_0}^{x_p} \frac{d\xi}{c(\xi,y)} \le \int_{1}^{x_p} \frac{d\xi}{c(\xi,y)}\,, \qquad \forall x_0 \in [-1,1]\,.$$

The left-hand side is negative and the right-hand side is positive, and there are two cases.

1. $0 < t_p < -\int_{-1}^{x_p} 1/c(\xi,y)\,d\xi$. Then since the integral is a continuous function in x_0, there exists $x_0 \in [-1,1]$ such that

   $$t_p = -\int_{x_0}^{x_p} \frac{d\xi}{c(\xi,y)}\,.$$

   From monotonicity the x_0 is unique, so $K(x_p,t_p,x_0,y) = L(x_p,t_p,x_0,0,y) = 0$ and the corresponding characteristics originate from the x-axis.

2. $t_p > -\int_{-1}^{x_p} 1/c(\xi,y)\,d\xi$. Then

   $$t_0 = t_p + \int_{-1}^{x_p} \frac{d\xi}{c(\xi,y)} > 0\,,$$

   so $K(x_p,t_p,-t_0-1,y) = L(x_p,t_p,-1,t_0,y) = 0$ and the corresponding characteristics originate from the boundary $x=-1$. Clearly $t_0 < t_p$, which is simply a restatement of causality.

When $c(x,y) > 0$, the argument is almost the same. □

From Lemmas 3.1 and 3.2, the characteristics equation $K(x,t,s,y) = 0$ implicitly defines a well-defined function $L : (-1,1)\times(0,T]\times[-1,1] \mapsto [-1-T, 1+T]$ — viz.

$$s = L(x,t,y)\,,$$

which maps every point in the strip of interest to a uniquely defined characteristic starting from $(f(s), g(s))$, and the solution representation is as follows:

Theorem 3.2. *There exists a unique solution $u(x,t,y)$ to the SPDE* (3.1)*, with the representation*

$$u(x,t,y) = h(L(x,t,y),y)\,.$$

3.3. Proof of the main Theorem 3.1

Let us now apply the characteristics approach to the SPDE (3.1) using the above solution representation, to find the expression for $\|u_y\|_{TV}$ and then suitable conditions to bound $\|u_y\|_{TV}$ in order to prove the main theorem.

From the solution representation,

$$\|u_y\|_{L^1} = \int_{-1}^{1} \rho(y) \int_{-1}^{1} \left| \frac{\partial h}{\partial s}\frac{\partial L}{\partial y} + \frac{\partial h}{\partial y} \right| dx\, dy\,.$$

We substitute the terms according to the different conditions, and evaluate the derivatives of L using the differential form of the implicit function theorem. Thus

$$\frac{\partial L}{\partial y} = -\frac{\partial K}{\partial y} \Big/ \frac{\partial K}{\partial s} = \begin{cases} D(-1,x)\,, & s < -1\,, \\ -c(s,y)D(s,x)\,, & s \in [-1,1]\,, \\ D(x,1)\,, & s > 1\,, \end{cases}$$

where

$$D(a,b) = \int_a^b \frac{c_y(\xi,y)}{c^2(\xi,y)}\, d\xi\,.$$

By direct calculation

$$\frac{\partial L}{\partial x} = -\frac{\partial K}{\partial x} \Big/ \frac{\partial K}{\partial s} = \begin{cases} -\dfrac{1}{c(x,y)}\,, & s < -1\,, \\ \dfrac{c(s,y)}{c(x,y)}\,, & s \in [-1,1]\,, \\ \dfrac{1}{c(x,y)}\,, & s > 1\,, \end{cases}$$

which is used later to calculate the Jacobian from dx to ds. Further, in order to change the variable of integration from dx to dt, introduce the function $x = M(t,s,y) \in (-1,1)$ implicitly defined by $K(x,t,s,y) = 0$ — i.e.

$$K(M(t,s,y),t,s,y) = 0\,.$$

This function basically traces the x coordinates of the characteristics s at time t, and is well-defined because $K(x,t,s,y)$ is monotonic in x.

We split the integral for $\|u_y\|_{TV}$ into three components according to the range of s:

$$\|u_y\|_{L^1} = \int_{-1}^{1} \rho(y) \left(\int_{\Omega_1} + \int_{\Omega_2} + \int_{\Omega_3} \right) \left| \frac{\partial h}{\partial s}\frac{\partial L}{\partial y} + \frac{\partial h}{\partial y} \right| dx\, dy\,,$$

where

$$\begin{aligned} \Omega_1 &= [-1,1] \cap \{x \mid s = L(x,t,y) < -1\}\,, \\ \Omega_2 &= [-1,1] \cap \{x \mid s = L(x,t,y) \in (-1,1)\}\,, \\ \Omega_3 &= [-1,1] \cap \{x \mid s = L(x,t,y) > 1\}\,. \end{aligned}$$

For the first integral, when $s < -1$ the characteristics originate from the left boundary $x = -1$ such that $c(x, y) < 0$, so

$$I_1 = \int_{-1}^{1} \rho(y) \int_{\Omega_1} \left| \frac{\partial h}{\partial s} D(-1, x) + \frac{\partial h}{\partial s} \right| dx\, dy$$
$$= \int_{c(-1,y)<0} \rho(y) \int_{\Omega_1} \left| \frac{\partial u_L}{\partial t}(-s-1, x) D(-1, x) + \frac{\partial u_L}{\partial y}(-s-1, y) \right| dx\, dy\,,$$

where $s = L(x, t, y)$. Then we change the variable $\theta = -s - 1 = -L(x, t, y) - 1 \in [0, T]$, and also have

$$s = L(x, t, y) \implies ds = \frac{\partial L}{\partial x} dx = -\frac{\partial K}{\partial x} / \frac{\partial K}{\partial s} dx = -\frac{1}{c(x, y)} dx\,,$$

where the partial derivative is used because t and y are kept constant. Thus the Jacobian of this change is given by

$$dx = -c(x, y)\, ds = c(x, y)\, d\theta\,.$$

Since the integrand is non-negative and $\theta \in [0, T]$, we can simply use $[0, T]$ as the range of integral, to obtain

$$I_1 \leq \int_{c(-1,y)<0} \rho(y) \int_{0}^{T} \left| \frac{\partial u_L}{\partial t}(\theta, x) D(-1, x) + \frac{\partial u_L}{\partial y}(\theta, y) \right| \left| c(x, y) \right| d\theta\, dy\,,$$

where $x = M(t, -\theta - 1, y)$ as before. For the third integral, we have similarly

$$I_3 = \int_{-1}^{1} \rho(y) \int_{\Omega_3} \left| \frac{\partial h}{\partial s} D(x, 1) + \frac{\partial h}{\partial s} \right| dx\, dy$$
$$\leq \int_{c(1,y)>0} \rho(y) \int_{0}^{T} \left| \frac{\partial u_R}{\partial t}(\theta, x) D(x, 1) + \frac{\partial u_L}{\partial y}(\theta, y) \right| \left| c(x, y) \right| d\theta\, dy\,.$$

We still have to evaluate the second integral, when the characteristics emanate from the x axis. Using the same technique with $\theta = s$ yields

$$I_2 = \int_{-1}^{1} \rho(y) \int_{\Omega_2} \left| \frac{\partial h}{\partial s}(-c(s, y) D(s, x)) + \frac{\partial h}{\partial s} \right| dx\, dy$$
$$\leq \int_{-1}^{1} \rho(y) \int_{-1}^{1} \left| -\frac{\partial u_0}{\partial t}(\theta, x) c(\theta, y) D(\theta, x) + \frac{\partial u_0}{\partial y}(\theta, y) \right| \left| \frac{c(x, y)}{c(\theta, y)} \right| d\theta\, dy$$
$$\leq \int_{-1}^{1} \rho(y) \int_{-1}^{1} \left[\left| \frac{\partial u_0}{\partial t}(\theta, x) D(\theta, x) \right| + \left| \frac{\partial u_0}{\partial y}(\theta, y) \frac{1}{c(\theta, y)} \right| \right] \left| c(x, y) \right| d\theta\, dy\,.$$

Finally, we define $\left|c(x,y)\right| \leq c_M(y)$ on $[-1,1]\times[-1,1]$ and substitute these bounds back into the expressions for I_1, I_2 and I_3. The conditions in the Theorem 3.1 ensure that the three integrals are bounded, and hence

$$\left\|u_y\right\|_{TV} = I_1 + I_2 + I_3 < \infty$$

so that the solution $u(x,t,y)$ is in BV.

4. Conclusions

The characteristics approach gives sharp estimates and leads to general theorems for proving regularity of the linear hyperbolic stochastic PDE. The approach outlined is conceptually straightforward. However, even when this approach is applied to a simple linear initial-boundary value problem, a rigorous proof still needs a careful discussion. In general, the main obstacle is the interplay between the geometry of the domain and the movement of the random characteristics, which makes an implicit representation of the solution difficult to obtain. For nonlinear cases, the solution will generally have shocks and it is even harder to find a solution representation of the solution. Nevertheless, it should be possible to extend the approach to nonlinear problems, for example by using the general characteristics described in Ref. [4].

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