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A Block Matrix Loop Algebra and Bi-Integrable Couplings of the Dirac Equations

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Abstract. A non-semisimple matrix loop algebra is presented, and a class of zero curvature equations over this loop algebra is used to generate bi-integrable couplings. An illustrative example is made for the Dirac soliton hierarchy. Associated variational identities yield bi-Hamiltonian structures of the resulting bi-integrable couplings, such that the hierarchy of bi-integrable couplings possesses infinitely many commuting symmetries and conserved functionals.

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1. Introduction

Zero curvature equations on semi-direct sums of loop algebras generate integrable couplings [1, 2], and the associated variational identities [3, 4] furnish Hamiltonian structures and bi-Hamiltonian structures of the resulting integrable couplings [5–7]. It is an important step in generating Hamiltonian structures to search for non-degenerate, symmetric and ad-invariant bilinear forms on the underlying loop algebras [8, 9]. Special semi-direct sums of loop algebras bring various interesting integrable couplings [8–12], including higher dimensional local bi-Hamiltonian ones [13–16] that greatly enrich multi-component integrable systems.

A zero curvature representation of a system of form

$$u_t = K(u) = K(x, t, u, u_x, u_{xx}, \cdots), \tag{1.1}$$

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where u is a column vector of dependent variables, means there exists a Lax pair [17] $U=U(u,\lambda)$ and $V=V(u,\lambda)$ belonging to a matrix loop algebra such that

$$U_t - V_x + [U,V] = 0 \tag{1.2}$$

generates the system [18]. An integrable coupling of the system (1.1) is an integrable system of the following form [13,14]:

$$\bar{u}_t = \bar{K}_1(\bar{u}) = \begin{bmatrix} K(u) \\ S(u,u_1) \end{bmatrix}, \qquad \bar{u} = \begin{bmatrix} u \\ u_1 \end{bmatrix}, \tag{1.3}$$

where u_1 is a new column vector of dependent variables. Further, an integrable coupling (1.3) is called nonlinear if the supplementary sub-vector field $S(u,u_1)$ is nonlinear with respect to the sub-vector u_1 [19,20], and an integrable system of the form

$$\bar{u}_t = \bar{K}(\bar{u}) = \begin{bmatrix} K(u) \\ S_1(u,u_1) \\ S_2(u,u_1,u_2) \end{bmatrix}, \qquad \bar{u} = \begin{bmatrix} u \\ u_1 \\ u_2 \end{bmatrix}, \tag{1.4}$$

is called a bi-integrable coupling of (1.1). In (1.4), it is notable that S_2 depends on the second column sub-vector u_2 but S_1 does not. We now proceed to use zero curvature equations in order to explore the generation of bi-integrable couplings and Hamiltonian structures for the resulting integrable couplings, through variational identities associated with enlarged Lax pairs.

One class of important integrable couplings consists of the so-called dark equations, motivated by the mysterious dark energy and dark matter envisaged in astronomy and cosmology [21]:

$$\begin{cases} u_t = K(x,t,u,u_x,u_{xx},\cdots), \\ \psi_t = A(u,\partial_x)\psi, \end{cases} \tag{1.5}$$

where $A(u,\partial_x)$ ia a linear differential operator. Dark energy is an hypothetical form of energy that is said to permeate all space, proposed to account for a missing part of the total mass in the entire universe (not in the form of visible stars and planets) in order to explain the observed increased rate of expansion of the universe [22,23]. Dark equations are linear extensions of the original system, which can extend the original equation further and further like general integrable couplings do. Importantly, they represent the large majority of integrable equations that we can readily consider, particularly in the study of integrable systems with multi-components, whereas nonlinear extensions are less amenable. In theory, they generalise the symmetry problem, and the first-order perturbation equations are special examples of dark equations with solutions that solve the original physical models to higher precision [13].

A soliton hierarchy is usually associated with a spectral problem

$$\phi_x = U\phi, \qquad U = U(u,\lambda) \in \tilde{\mathfrak{g}}, \tag{1.6}$$

where λ is a spectral parameter and $\tilde{\mathfrak{g}}$ is a matrix loop algebra associated with a given matrix Lie algebra $\mathfrak{g}$, often simple. Let us assume that the corresponding stationary zero curvature equation

$$W_x = [U, W] \tag{1.7}$$

has a solution of the form

$$W = W(u, \lambda) = \sum_{i \geq 0} W_{0,i} \lambda^{-i}, \tag{1.8}$$

where $W_{0,i} \in \mathfrak{g},\ i \geq 0$. We introduce a sequence of temporal spectral problems

$$\phi_{t_m} = V^{[m]} \phi = V^{[m]}(u, \lambda)\phi, \quad m \geq 0, \tag{1.9}$$

involving the Lax matrices defined by

$$V^{[m]} = (\lambda^m W)_+ + \Delta_m \in \tilde{\mathfrak{g}}, \quad m \geq 0, \tag{1.10}$$

where Q_+ denotes the polynomial part of Q in λ and the modification terms Δ_m are intended to guarantee that the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0, \tag{1.11}$$

generate a soliton hierarchy with Hamiltonian structures — viz.

$$u_{t_m} = K_m(u) = J \frac{\delta \mathscr{H}_m}{\delta u}, \quad m \geq 0. \tag{1.12}$$

The Hamiltonian functionals $\mathscr{H}_m$ are generally furnished by applying the variational identity [3,9]

$$\frac{\delta}{\delta u} \int \langle W, U_\lambda \rangle \, dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \langle W, U_u \rangle, \tag{1.13}$$

where W is a solution of (1.7), $U_\lambda = \partial U / \partial \lambda$ and $U_u = \partial U / \partial u$, the constant γ is determined by

$$\gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} \ln |\langle W, W \rangle|, \tag{1.14}$$

and $\langle \cdot, \cdot \rangle$ is a bilinear form on the matrix loop algebra $\tilde{\mathfrak{g}}$ that is non-degenerate, symmetric and ad-invariant [9]. The soliton hierarchy (1.12) has the commutativity properties:

$$[K_m, K_n] := K_m'(u)[K_n] - K_n'(u)[K_m] = 0, \tag{1.15}$$

$$\{\mathscr{H}_m, \mathscr{H}_n\}_J := \int \left(\frac{\delta \mathscr{H}_m}{\delta u} \right)^T J \frac{\delta \mathscr{H}_n}{\delta u} \, dx = 0, \tag{1.16}$$

where $m, n \geq 0$. These properties imply that the hierarchy (1.12) possesses infinitely many commuting symmetries $\{K_n\}_{n=0}^\infty$ and conserved functionals $\{\mathscr{H}_n\}_{n=0}^\infty$.

The rest of this article is structured as follows. In Section 2, a non-semisimple matrix loop algebra consisting of 3×3 block matrices is introduced, and used as a starting point to formulate a practical way of constructing integrable Hamiltonian couplings of given integrable systems. In Section 3, an application to the Dirac soliton hierarchy is discussed, and nonlinear bi-integrable couplings are constructed for the equations derived from a spectral problem posed by Dirac's equation for relativistic spinors in a spherical symmetric potential. The corresponding variational identity furnishes Hamiltonian structures for the resulting integrable couplings. While generating these Hamiltonian structures, a crucial step is to search for non-degenerate, symmetric and ad-invariant bilinear forms on the underlying loop algebra. In the final section, a few concluding remarks are made.

2. Matrix Loop Algebra yielding Bi-Integrable Couplings

2.1. Matrix loop algebra

Let us fix an arbitrary non-zero constant α from a given field of numbers Γ. To construct bi-integrable couplings, we introduce triangular block matrices

$$M(A_1,A_2,A_3) = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 & \alpha A_2 \\ 0 & 0 & A_1 \end{bmatrix}, \tag{2.1}$$

where A_1,A_2 and A_3 are arbitrary square matrices of the same order. Obviously, the matrix product of two such block matrices may be written

$$M(A_1,A_2,A_3)M(B_1,B_2,B_3) = M(C_1,C_2,C_3)\,, \tag{2.2}$$

where

$$\begin{cases} C_1 = A_1B_1\,, \\ C_2 = A_1B_2 + A_2B_1\,, \\ C_3 = A_1B_3 + A_3B_1 + \alpha A_2B_2\,. \end{cases} \tag{2.3}$$

Such an essential closure property under matrix multiplication ensures that all block matrices defined above constitute a Lie algebra over the field Γ (e.g. see [24]), under the matrix commutator

$$[M_1,M_2] = M_1M_2 - M_2M_1\,. \tag{2.4}$$

The associated loop matrix algebra $\bar{\mathfrak{g}}(\lambda)$ is formed by all block matrices of the type (2.1) — i.e.

$$\bar{\mathfrak{g}}(\lambda) = \{M(A_1,A_2,A_3)\,|\,M \text{ defined by (2.1), entries of } A_i \text{ Laurent series in } \lambda\}, \tag{2.5}$$

and its Lie bracket is defined by (2.4). The loop algebra $\bar{\mathfrak{g}}(\lambda)$ is non-semisimple, due to the semi-direct sum decomposition

$$\bar{\mathfrak{g}}(\lambda) = \tilde{\mathfrak{g}} \Subset \tilde{\mathfrak{g}}_c\,, \tag{2.6}$$

where $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}_c$ are two loop subalgebras of $\bar{\mathfrak{g}}(\lambda)$ — viz.

$$\begin{cases} \tilde{\mathfrak{g}} = \{M(A_1,0,0)\,|\,\text{entries of } A_1 \text{ Laurent series in } \lambda\}\,, \\ \tilde{\mathfrak{g}}_c = \{M(0,A_2,A_3)\,|\,\text{entries of } A_2, A_3 \text{ Laurent series in } \lambda\}\,. \end{cases}$$

It follows directly from the semi-direct sum property that $\tilde{\mathfrak{g}}_c$ is a nontrivial ideal Lie subalgebra of $\bar{\mathfrak{g}}(\lambda)$.

This matrix loop algebra $\bar{\mathfrak{g}}(\lambda)$ provides an underlying mathematical structure for formulating nonlinear bi-integrable Hamiltonian couplings, in a similar way to the one proposed in Ref. [25]. The first matrix block A_1 corresponds to the initial integrable system as required, and the second and third matrix blocks A_2 and A_3 generate the supplementary sub-vector fields S_1 and S_2 in (1.4). The sub-commutators $[A_2,B_2]$, $[A_2,B_3]$ and $[A_3,B_2]$ generally engender nonlinear terms in the resulting systems of bi-integrable couplings.

2.2. Bi-integrable couplings

Let us introduce an enlarged spectral matrix $\bar{U}$ from the above matrix loop algebra $\bar{\mathfrak{g}}(\lambda)$ — i.e.

$$\bar{U} = \bar{U}(\bar{u},\lambda) = M(U,U_1,U_2) = M(U(u,\lambda),U_1(u_1,\lambda),U_2(u_2,\lambda))\,, \tag{2.7}$$

where $\bar{u}$ consists of three column sub-vectors of dependent variables u, u_1 and u_2 defined as in (1.4). Then an enlarged zero curvature equation

$$\bar{U}_t - \bar{V}_x + [\bar{U},\bar{V}] = 0 \tag{2.8}$$

with an enlarged Lax matrix $\bar{V}$ from $\bar{\mathfrak{g}}(\lambda)$,

$$\bar{V} = \bar{V}(\bar{u},\lambda) = M(V,V_1,V_2) = M(V(u,\lambda),V_1(u,u_1,\lambda),V_2(u,u_1,u_2,\lambda))\,, \tag{2.9}$$

yields the following triangular system:

$$\begin{cases} U_t - V_x + [U,V] = 0, \\ U_{1,t} - V_{1,x} + [U,V_1] + [U_1,V] = 0\,, \\ U_{2,t} - V_{2,x} + [U,V_2] + [U_2,V] + \alpha[U_1,V_1] = 0\,. \end{cases} \tag{2.10}$$

From the zero curvature representation (1.2) of the system (1.1), this produces a bi-integrable coupling of (1.1) that is usually nonlinear with respect to the supplementary variables u_1 and u_2, and thus candidates for nonlinear bi-integrable couplings.

To generate a hierarchy of integrable couplings of (1.1), given $\bar{\mathfrak{g}}(\lambda)$ we seek a solution

$$\bar{W} = \bar{W}(\bar{u},\lambda) = M(W,W_1,W_2) = M(W(u,\lambda),W_1(u,u_1,\lambda),W_2(u,u_1,u_2,\lambda)) \tag{2.11}$$

to the enlarged stationary zero curvature equation

$$\bar{W}_x = [\bar{U},\bar{W}]\,, \tag{2.12}$$

equivalent to requiring

$$
\begin{cases}
W_x = [U, W]\,, \\
W_{1,x} = [U, W_1] + [U_1, W]\,, \\
W_{2,x} = [U, W_2] + [U_2, W] + \alpha[U_1, W_1]\,.
\end{cases}
\tag{2.13}
$$

We can often obtain a solution of the type (e.g. see [18, 26])

$$
W_1 = \sum_{i=0}^{\infty} W_{1,i}\lambda^{-i}\,, \qquad W_2 = \sum_{i=0}^{\infty} W_{2,i}\lambda^{-i}\,,
\tag{2.14}
$$

together with a solution for W defined by (1.8). We then take a set of enlarged matrix modifications $\bar{\Delta}_m \in \bar{\mathfrak{g}}(\lambda)$, $m \geq 0$, and introduce the enlarged Lax matrices

$$
\bar{V}^{[m]} = (\lambda^m \bar{W})_+ + \bar{\Delta}_m\,, \quad m \geq 0\,,
\tag{2.15}
$$

where the subscript "+" denotes the polynomial part, such that the enlarged zero curvature equations

$$
\bar{U}_{t_m} - \bar{V}_x^{[m]} + [\bar{U}, \bar{V}^{[m]}] = 0\,, \quad m \geq 0\,,
\tag{2.16}
$$

engender a soliton hierarchy of bi-integrable couplings for the system (1.1). Generally, integrable couplings obtained this way commute with each other, and therefore form infinitely many common symmetries for the whole hierarchy of integrable couplings.

Hamiltonian structures of such bi-integrable couplings can be constructed through using the associated variational identities [3, 9], including the trace identity [18] and the component-trace identity [10] as particular examples. A crucial step is to find bilinear forms over the underlying loop algebra that should satisfy the non-degenerate property, the symmetric property and the ad-invariant property. Then on applying the corresponding variational identity

$$
\frac{\delta}{\delta \bar{u}} \int \langle \bar{W}, \bar{U}_\lambda \rangle \, dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \langle \bar{W}, \bar{U}_{\bar{u}} \rangle\,,
\tag{2.17}
$$

with the constant γ determined by

$$
\gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} \ln|\langle \bar{W}, \bar{W} \rangle|\,,
\tag{2.18}
$$

we can generate Hamiltonian structures for the resulting bi-integrable couplings. In this variational identity, $\langle \cdot, \cdot \rangle$ denotes a required non-degenerate, symmetric and ad-invariant bilinear form over the underlying loop algebra consisting of square matrices of the form (2.7) — cf. [3, 8, 10] for details. The Hamiltonian structures link the symmetries and conservation laws together, and thus ensure the existence of infinitely many conservation laws.

In the next section, an illustrative example is produced by applying the above general computational paradigm, and nonlinear Hamiltonian bi-integrable couplings are then computed for the Dirac equations.

3. Bi-Integrable Couplings of the Dirac Equations

3.1. The Dirac soliton hierarchy

Let us recall the Dirac soliton hierarchy [27], for which the well-known spectral problem is

$$\phi_x = U\phi\ , \qquad U = U(u,\lambda) = \begin{bmatrix} p & \lambda+q \\ -\lambda+q & -p \end{bmatrix} \in \widetilde{\mathrm{sl}}(2)\ , \tag{3.1}$$

where

$$u = \begin{bmatrix} p \\ q \end{bmatrix}, \qquad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}\ ,$$

and $\widetilde{\mathrm{sl}}(2)$ is the special matrix loop algebra — i.e.

$$\tilde{\mathfrak{g}} = \widetilde{\mathrm{sl}}(2) = \{A \in \mathrm{sl}(2)\,|\,\text{entries of } A \text{ Laurent series in } \lambda\}\ . \tag{3.2}$$

The stationary zero curvature equation

$$W_x = [U,W] \tag{3.3}$$

determines

$$\begin{cases} a_x = -2\lambda\, c + 2pb\ , \\ b_x = -2qc + 2pa\ , \\ c_x = 2\lambda\, a - 2qb\ , \end{cases} \tag{3.4}$$

if we assume that W is of the form

$$W = \begin{bmatrix} c & a+b \\ a-b & -c \end{bmatrix} = \sum_{i\geq 0} W_{0,i}\lambda^{-i} = \sum_{i\geq 0} \begin{bmatrix} c_i & a_i+b_i \\ a_i-b_i & -c_i \end{bmatrix} \lambda^{-i} \in \widetilde{\mathrm{sl}}(2)\ . \tag{3.5}$$

The system (3.4) equivalently generates

$$\begin{cases} a_{i+1} = \dfrac{1}{2}c_{i,x} + qb_i\ , \\ c_{i+1} = -\dfrac{1}{2}a_{i,x} + pb_i\ , \\ b_{i+1,x} = -2qc_{i+1} + 2pa_{i+1}\ , \end{cases} \qquad i \geq 0\ . \tag{3.6}$$

So assuming the initial values

$$b_0 = -1\ , \quad a_0 = c_0 = 0\ , \tag{3.7}$$

and that the constant of integration as zero, we can work out the first few sets:

$$\begin{aligned} &a_1 = -q\ , && c_1 = -p\ , && b_1 = 0\ , \\ &a_2 = -\frac{1}{2}p_x\ , && c_2 = \frac{1}{2}q_x\ , && b_2 = -\frac{1}{2}p^2 - \frac{1}{2}q^2\ , \\ &a_3 = \frac{1}{4}q_{xx} - \frac{1}{2}p^2q - \frac{1}{2}q^3\ , && c_3 = \frac{1}{4}p_{xx} - \frac{1}{2}p^3 - \frac{1}{2}pq^2\ , && b_3 = \frac{1}{2}(pq_x - p_xq)\ . \end{aligned}$$

The zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0 \ \text{ with } \ V^{[m]} = (\lambda^m W)_+ \tag{3.8}$$

where $m \geq 0$, then provide the well-known Dirac hierarchy of soliton equations — viz.

$$u_{t_m} = K_m = \begin{bmatrix} 2a_{m+1} \\ -2c_{m+1} \end{bmatrix} = \Phi^m \begin{bmatrix} -2q \\ 2p \end{bmatrix} = J \frac{\delta \mathscr{H}_m}{\delta u} \tag{3.9}$$

where $m \geq 0$, and the Hamiltonian operator J, the hereditary recursion operator Φ and the Hamiltonian functions are defined as follows:

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ \Phi = \begin{bmatrix} 2q\partial^{-1}p & -\frac{1}{2}\partial + 2q\partial^{-1}q \\ \frac{1}{2}\partial - 2p\partial^{-1}p & -2p\partial^{-1}q \end{bmatrix}, \ \mathscr{H}_m = \int \frac{2b_{m+2}}{m+1} \, dx \,, \tag{3.10}$$

in which $m \geq 0$, and $\partial = \partial / \partial x$.

3.2. Hamiltonian bi-integrable couplings

We now proceed to construct Hamiltonian bi-integrable couplings for the Dirac soliton hierarchy (3.9). An enlarged spectral matrix from $\bar{\mathfrak{g}}(\lambda)$ defined by (2.5) is chosen as

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = M(U, U_1, U_2) \,, \qquad \bar{u} = (p, q, r, s, v, w)^T \,, \tag{3.11}$$

where U is defined as in (3.1) and the supplementary spectral matrices U_1 and U_2 are given by

$$U_1 = U_1(u_1) = \begin{bmatrix} r & s \\ s & -r \end{bmatrix} \in \widetilde{\mathrm{sl}}(2) \,, \quad u_1 = \begin{bmatrix} r \\ s \end{bmatrix} , \tag{3.12}$$

$$U_2 = U_2(u_2) = \begin{bmatrix} v & w \\ w & -v \end{bmatrix} \in \widetilde{\mathrm{sl}}(2) \,, \quad u_2 = \begin{bmatrix} v \\ w \end{bmatrix} . \tag{3.13}$$

As usual, in order to solve the enlarged stationary zero curvature equation (2.12), we take a solution of the following type:

$$\bar{W} = \bar{W}(\bar{u}, \lambda) = M(W, W_1, W_2) \in \bar{\mathfrak{g}}(\lambda) \,, \tag{3.14}$$

where W defined by (3.3) solves $W_x = [U, W]$, and W_1 and W_2 are assumed to be

$$W_1 = W_1(u, u_1, \lambda) = \begin{bmatrix} g & e + f \\ e - f & -g \end{bmatrix} \in \widetilde{\mathrm{sl}}(2) \,, \tag{3.15}$$

$$W_2 = W_2(u, u_1, u_2, \lambda) = \begin{bmatrix} g' & e' + f' \\ e' - f' & -g' \end{bmatrix} \in \widetilde{\mathrm{sl}}(2) \,. \tag{3.16}$$

The second and third equations in (2.13) then equivalently generate

$$\begin{cases} e_x = 2pf - 2\lambda g + 2rb\,, \\ f_x = 2pe - 2qg + 2ra - 2sc\,, \\ g_x = 2\lambda e - 2qf - 2sb\,, \end{cases} \tag{3.17}$$

and

$$\begin{cases} e'_x = 2pf' - 2\lambda g' + 2vb + 2\alpha rf\,, \\ f'_x = 2pe' - 2qg' + 2va - 2wc + 2\alpha re - 2\alpha sg\,, \\ g'_x = 2\lambda e' - 2qf' - 2wb - 2\alpha sg\,, \end{cases} \tag{3.18}$$

respectively. Trying a formal series solution $\bar{W}$ by setting

$$\begin{cases} e = \sum_{i=0}^{\infty} e_i\lambda^{-i}, \quad f = \sum_{i=0}^{\infty} f_i\lambda^{-i}, \quad g = \sum_{i=0}^{\infty} g_i\lambda^{-i}, \\ e' = \sum_{i=0}^{\infty} e'_i\lambda^{-i}, \quad f' = \sum_{i=0}^{\infty} f'_i\lambda^{-i}, \quad g' = \sum_{i=0}^{\infty} g'_i\lambda^{-i}, \end{cases} \tag{3.19}$$

we arrive at

$$\begin{cases} e_{i+1} &= \dfrac{1}{2} g_{i,x} + qf_i + sb_i\,, \\ g_{i+1} &= -\dfrac{1}{2} e_{i,x} + pf_i + rb_i\,, \\ f_{i+1,x} &= 2pe_{i+1} - 2qg_{i+1} + 2ra_{i+1} - 2sc_{i+1}\,, \\ e'_{i+1} &= \dfrac{1}{2} g'_{i,x} + qf'_i + wb_i + \alpha sf_i\,, \\ g'_{i+1} &= -\dfrac{1}{2} e'_{i,x} + pf'_i + vb_i + \alpha rf_i\,, \\ f'_{i+1,x} &= 2pe'_{i+1} - 2qg'_{i+1} + 2va_{i+1} - 2wc_{i+1} \\ & \quad -2\alpha re_{i+1} - 2\alpha sg_{i+1}\,, \end{cases} \tag{3.20}$$

where $i \geq 0$. We take initial data as

$$f_0 = -1\,, \quad e_0 = g_0 = 0\,; \quad f'_0 = -1\,, \quad e'_0 = g'_0 = 0\,; \tag{3.21}$$

and suppose that

$$\begin{cases} e_i|_{\bar{u}=0} = f_i|_{\bar{u}=0} = g_i|_{\bar{u}=0} = 0\,, \quad i \geq 1\,, \\ e'_i|_{\bar{u}=0} = f'_i|_{\bar{u}=0} = g'_i|_{\bar{u}=0} = 0\,, \quad i \geq 1\,. \end{cases} \tag{3.22}$$

Then the recursion relation (3.20) uniquely generates the sequence of e_i, f_i, g_i and e_i', f_i', g_i', $i \geq 1$, recursively. From the recursion relation, one therefore obtains

$$\left\{\begin{aligned} e_1 &= -q - s\,,\\ g_1 &= -p - r\,,\\ f_1 &= 0\,;\end{aligned}\right.$$

$$\left\{\begin{aligned} e_2 &= -\frac{1}{2}(p_x + r_x)\,,\\ g_2 &= \frac{1}{2}(q_x + s_x)\,,\\ f_2 &= -\frac{1}{2}(p^2 + q^2) - (pr + qs)\,;\end{aligned}\right.$$

$$\left\{\begin{aligned} e_3 &= \frac{1}{4}(q_{xx} + s_{xx}) - \frac{1}{2}(p^2 + q^2)(q + s) - (pr + qs)q\,,\\ g_3 &= \frac{1}{4}(p_{xx} + r_{xx}) - \frac{1}{2}(p^2 + q^2)(p + r) - (pr + qs)p\,,\\ f_3 &= \frac{1}{2}(pq_x + ps_x - p_x q - p_x s - qr_x + q_x r)\,;\end{aligned}\right.$$

and

$$\left\{\begin{aligned} e_1' &= -q - w - \alpha s\,,\\ g_1' &= -p - v - \alpha r\,,\\ f_1' &= 0\,;\end{aligned}\right.$$

$$\left\{\begin{aligned} e_2' &= -\frac{1}{2}(p_x + v_x + \alpha r_x)\,,\\ g_2' &= \frac{1}{2}(q_x + w_x + \alpha s_x)\,,\\ f_2' &= -\frac{1}{2}(p^2 + q^2) - \frac{1}{2}\alpha(r^2 + s^2) - (\alpha r + v)p - (\alpha s + w)q\,;\end{aligned}\right.$$

$$\left\{\begin{aligned} e_3' &= \frac{1}{4}(q_{xx} + \alpha s_{xx} + w_{xx}) - \frac{1}{2}(q + w + \alpha s)(p^2 + q^2)\\ &\quad -\frac{1}{2}\alpha(r^2 + s^2)q - (qw + pv)q - \alpha(pr + qs)(q + s)\,,\\ g_3' &= \frac{1}{4}(p_{xx} + \alpha v_{xx} + r_{xx}) - \frac{1}{2}(p + v + \alpha r)(p^2 + q^2)\\ &\quad -\frac{1}{2}\alpha(r^2 + s^2)p - (qw + pv)p - \alpha(pr + qs)(p + r)\,,\\ f_3' &= \frac{1}{2}[-(q + \alpha s + w)p_x + (p + \alpha r + v)q_x\\ &\quad -\alpha(q + s)r_x + \alpha(p + r)s_x - qv_x + pw_x]\,.\end{aligned}\right.$$

The above functions are all differential polynomials in the six variables p, q, r, s, v, w.

For each integer $m \geq 0$, let us further introduce an enlarged Lax matrix

$$\bar{V}^{[m]} = (\lambda^m \bar{W})_+ = M(V^{[m]}, V_1^{[m]}, V_2^{[m]}) \in \bar{\mathfrak{g}}(\lambda)\,, \tag{3.23}$$

where $V^{[m]}$ is defined as in (3.8) and $V_i^{[m]} = (\lambda^m W_i)_+ \in \widetilde{\mathrm{sl}}(2)$, $i = 1, 2$, when the enlarged zero curvature equation

$$\bar{U}_{t_m} - (\bar{V}^{[m]})_x + [\bar{U}, \bar{V}^{[m]}] = 0 \tag{3.24}$$

yields

$$\begin{cases} U_{1,t_m} - V_{1,x}^{[m]} + [U, V_1^{[m]}] + [U_1, V^{[m]}] = 0\,, \\ U_{2,t_m} - V_{2,x}^{[m]} + [U, V_2^{[m]}] + [U_2, V^{[m]}] + \alpha[U_1, V_1^{[m]}] = 0\,, \end{cases}$$

together with the m-th Dirac system in (3.9). The above two equations then present the supplementary systems

$$\bar{v}_{t_m} = S_m = S_m(\bar{u}) = \begin{bmatrix} S_{1,m}(u, u_1) \\ S_{2,m}(u, u_1, u_2) \end{bmatrix}, \quad m \geq 0\,, \tag{3.25}$$

where $\bar{v} = (r, s, v, w)^T$ and

$$S_{1,m}(u, u_1) = \begin{bmatrix} 2e_{m+1} \\ -2g_{m+1} \end{bmatrix},$$

$$S_{2,m}(u, u_1, u_2) = \begin{bmatrix} 2e'_{m+1} \\ -2g'_{m+1} \end{bmatrix}.$$

The enlarged zero curvature equations thus generate a hierarchy of bi-integrable couplings:

$$\bar{u}_{t_m} = \begin{bmatrix} p \\ q \\ r \\ s \\ v \\ w \end{bmatrix}_{t_m} = \bar{K}_m(\bar{u}) = \begin{bmatrix} 2a_{m+1} \\ -2c_{m+1} \\ 2e_{m+1} \\ -2g_{m+1} \\ 2e'_{m+1} \\ -2g'_{m+1} \end{bmatrix}, \quad m \geq 0 \tag{3.26}$$

for the Dirac soliton hierarchy (3.9).

Except the first two, all bi-integrable couplings defined by (3.26) are nonlinear, since the supplementary systems (3.25) with $m \geq 2$ are nonlinear with respect to the four dependent variables r, s, v, w. This implies that (3.26) provides a hierarchy of nonlinear bi-integrable couplings for the Dirac soliton hierarchy. The first nonlinear bi-integrable coupling system reads

$$\begin{cases} p_{t_2} = 2a_3\,,\ q_{t_2} = -2c_3\,,\ r_{t_2} = 2e_3\,, \\ s_{t_2} = -2g_3\,,\ v_{t_2} = 2e'_3\,,\ w_{t_2} = -2g'_3\,, \end{cases} \tag{3.27}$$

where b_3, c_3, f_3, g_3, f'_3, g'_3 are defined as before.

3.3. Hamiltonian structures

In order to generate Hamiltonian structures for the resulting bi-integrable couplings in (3.26), we apply the following variational identity over the enlarged matrix loop algebra $\bar{\mathfrak{g}}(\lambda)$ [8,9]:

$$\frac{\delta}{\delta \bar{u}} \int \langle \bar{W}, \bar{U}_\lambda \rangle \, dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \langle \bar{W}, \bar{U}_{\bar{u}} \rangle \,, \quad \gamma = \text{constant} \,. \tag{3.28}$$

To construct symmetric and ad-invariant bilinear forms on $\bar{\mathfrak{g}}(\lambda)$ conveniently, we first transform the semi-direct sum $\bar{\mathfrak{g}}(\lambda)$ into vector form. Define a mapping

$$\sigma : \bar{\mathfrak{g}}(\lambda) \to \mathbb{R}^9, \, A \mapsto (a_1, \cdots, a_9)^T \,, \tag{3.29}$$

where

$$A = M(A_1, A_2, A_3) \in \bar{\mathfrak{g}}(\lambda) \,, \; A_i = \begin{bmatrix} a_{3i} & a_{3i-2} + a_{3i-1} \\ a_{3i-2} - a_{3i-1} & -a_{3i} \end{bmatrix}, \; 1 \le i \le 3 \,. \tag{3.30}$$

As usual, this mapping σ induces a Lie algebraic structure on $\mathbb{R}^9$, isomorphic to the enlarged matrix loop algebra $\bar{\mathfrak{g}}(\lambda)$. The corresponding Lie bracket $[\cdot,\cdot]$ on $\mathbb{R}^9$ can be computed as

$$[a, b]^T = a^T R(b) \,, \tag{3.31}$$

where $a = (a_1, \cdots, a_9)^T, \; b = (b_1, \cdots, b_9)^T \in \mathbb{R}^9$ and

$$R(b) = M(R_1, R_2, R_3) \,, \tag{3.32}$$

with

$$R_i = \begin{bmatrix} 0 & -2b_{3i} & -2b_{3i-1} \\ -2b_{3i} & 0 & 2b_{3i-2} \\ 2b_{3i-1} & 2b_{3i-2} & 0 \end{bmatrix}, \quad 1 \le i \le 3 \,.$$

This Lie algebra $(\mathbb{R}^9, [\cdot,\cdot])$ is isomorphic to the enlraged matrix loop algebra $\bar{\mathfrak{g}}(\lambda)$ defined in Section 2, and the mapping σ defined by (3.29) is a Lie isomorphism between those two Lie algebras.

A bilinear form on $\mathbb{R}^9$ can be provided by

$$\langle a, b \rangle = a^T F b \,, \tag{3.33}$$

where F is a constant matrix (actually, $F = (\langle \mathbf{e}_i, \mathbf{e}_j \rangle)_{9\times 9}$, where $\mathbf{e}_1, \cdots, \mathbf{e}_9$ are the standard basis of $\mathbb{R}^9$). The symmetric property $\langle a, b \rangle = \langle b, a \rangle$ requires that

$$F^T = F \,. \tag{3.34}$$

Under this symmetric condition, the ad-invariance property

$$\langle a, [b, c] \rangle = \langle [a, b], c \rangle$$

requires that

$$F(R(b))^T = -R(b)F\,, \quad b \in \mathbb{R}^9\,. \tag{3.35}$$

This matrix equation involves an arbitrary vector b, and thus leads to a linear system of equations on the elements of the matrix F. On solving the resulting system, we obtain

$$F = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \alpha\eta_3 & 0 \\ \eta_3 & 0 & 0 \end{bmatrix} \otimes F_0 \tag{3.36}$$

with

$$F_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where η_i, $1 \le i \le 3$, are arbitrary constants. The corresponding bilinear form on the semi-direct sum $\bar{\mathfrak{g}}(\lambda)$ of the two Lie subalgebras $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}_c$ is then defined as

$$\begin{aligned} \langle A,B\rangle_{\bar{\mathfrak{g}}(\lambda)} =& \langle \sigma(A),\sigma(B)\rangle_{\mathbb{R}^9} \\ =&(a_1,\cdots,a_9)F(b_1,\cdots,b_9)^T \\ =&(a_1b_1 - a_2b_2 + a_3b_3)\eta_1 + (a_1b_4 - a_2b_5 + a_3b_6 + a_4b_1 - a_5b_2 + a_6b_3)\eta_2 \\ &+ (a_1b_7 - a_2b_8 + a_3b_9 + \alpha a_4b_4 - \alpha a_5b_5 + \alpha a_6b_6 + a_7b_1 - a_8b_2 + a_9b_3)\eta_3, \end{aligned} \tag{3.37}$$

where A and B are two matrices in $\bar{\mathfrak{g}}(\lambda)$ — viz.

$$\begin{cases} A = \sigma^{-1}((a_1,\cdots,a_9)^T) \in \bar{\mathfrak{g}}(\lambda)\,, \\ B = \sigma^{-1}((b_1,\cdots,b_9)^T) \in \bar{\mathfrak{g}}(\lambda)\,. \end{cases} \tag{3.38}$$

Due to the isomorphism of σ, the bilinear form (3.37) is also symmetric and ad-invariant — i.e.

$$\langle A,B\rangle_{\bar{\mathfrak{g}}(\lambda)} = \langle B,A\rangle_{\bar{\mathfrak{g}}(\lambda)}\,, \quad \langle A,[B,C]\rangle_{\bar{\mathfrak{g}}(\lambda)} = \langle [A,B],C\rangle_{\bar{\mathfrak{g}}(\lambda)}\,,$$

where A, B, $C \in \bar{\mathfrak{g}}(\lambda)$. However, this kind of bilinear form is not of Killing type, since the enlarged matrix loop algebra $\bar{\mathfrak{g}}(\lambda)$ is not semisimple. A bilinear form, defined by (3.37), is non-degenerate if and only if the determinant of F is not zero — i.e.

$$\det(F) = {\eta_3}^9\alpha^3 \ne 0\,. \tag{3.39}$$

We can therefore choose η_1, η_2 and η_3 such that $\det(F)$ is non-zero, to obtain non-degenerate bilinear forms over the enlarged matrix loop algebra $\bar{\mathfrak{g}}(\lambda)$.

We can now compute

$$\langle \bar{W}, \bar{U}_\lambda \rangle_{\bar{\mathfrak{g}}(\lambda)} = -b\,\eta_1 - f\,\eta_2 - f'\,\eta_3 \,,$$

and

$$\langle \bar{W}, \bar{U}_{\bar{u}} \rangle_{\bar{\mathfrak{g}}(\lambda)} = \begin{bmatrix} c\,\eta_1 + g\,\eta_2 + g'\,\eta_3 \\ a\,\eta_1 + e\,\eta_2 + e'\,\eta_3 \\ c\,\eta_2 + \alpha\, g\,\eta_3 \\ a\,\eta_2 + \alpha\, e\,\eta_3 \\ c\,\eta_3 \\ a\,\eta_3 \end{bmatrix} .$$

Moreover, the formula (2.18) for the constant γ directly yields $\gamma = 0$, and so the corresponding variational identity is

$$\frac{\delta}{\delta \bar{u}} \int \frac{b_{m+1}\,\eta_1 + f_{m+1}\,\eta_2 + f'_{m+1}\,\eta_3}{m}\, dx = \begin{bmatrix} c_m\,\eta_1 + g_m\,\eta_2 + g'_m\,\eta_3 \\ a_m\,\eta_1 + e_m\,\eta_2 + e'_m\,\eta_3 \\ c_m\,\eta_2 + \alpha\, g_m\,\eta_3 \\ a_m\,\eta_2 + \alpha\, e_m\,\eta_3 \\ c_m\,\eta_3 \\ a_m\,\eta_3 \end{bmatrix}, \quad m \geq 1 \,.$$

Consequently, we obtain a Hamiltonian structure for the hierarchy (3.26) of bi-integrable couplings — viz.

$$\bar{u}_{t_m} = \bar{J}\,\frac{\delta \bar{\mathscr{H}}_m}{\delta \bar{u}}\,, \quad m \geq 0\,, \tag{3.40}$$

with the Hamiltonian functionals

$$\bar{\mathscr{H}}_m = \int \frac{2\,b_{m+2}\,\eta_1 + 2\,f_{m+2}\,\eta_2 + 2 f'_{m+2}\,\eta_3}{m+1}\, dx\,, \tag{3.41}$$

and the Hamiltonian operator

$$\bar{J} = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \alpha\,\eta_3 & 0 \\ \eta_3 & 0 & 0 \end{bmatrix}^{-1} \otimes J\,, \tag{3.42}$$

where J is defined as in (3.10). In passing, we remark that a condition in defining $\bar{J}$ is that $\det(F) \neq 0$.

3.4. Symmetries and conserved functionals

Checking the recursion relation

$$\bar{K}_m = \bar{\Phi}\bar{K}_{m-1}\,, \quad m \geq 0\,, \tag{3.43}$$

we have a recursion operator $\bar{\Phi}$ (cf. [28])

$$\bar{\Phi} = M^T(\Phi, \Phi_1, \Phi_2) \tag{3.44}$$

where Φ is given by (3.10) and

$$\Phi_1 = \begin{bmatrix} 2q\partial^{-1}r + 2s\partial^{-1}p & 2q\partial^{-1}s + 2s\partial^{-1}q \\ -2p\partial^{-1}r - 2r\partial^{-1}p & -2p\partial^{-1}s - 2r\partial^{-1}q \end{bmatrix}, \tag{3.45}$$

$$\Phi_2 = \begin{bmatrix} 2q\partial^{-1}r + 2w\partial^{-1}p + 2\alpha s\partial^{-1}r & 2q\partial^{-1}w + 2w\partial^{-1}q + 2\alpha s\partial^{-1}s \\ -2p\partial^{-1}v - 2v\partial^{-1}p - 2\alpha r\partial^{-1}r & -2p\partial^{-1}w - 2v\partial^{-1}q - 2\alpha r\partial^{-1}s \end{bmatrix}. \tag{3.46}$$

One may show by computer algebra systems that $\bar{\Phi}$ is a hereditary operator [29,30]. Thus the quantity

$$\bar{\Phi}'(\bar{u})[\bar{\Phi}\bar{T}_1]\bar{T}_2 - \bar{\Phi}\bar{\Phi}'(\bar{u})[\bar{T}_1]\bar{T}_2$$

is symmetric with respect to $\bar{T}_1$ and $\bar{T}_2$, and the two operators $\bar{J}$ and $\bar{M} = \bar{\Phi}\bar{J}$ constitute a Hamiltonian pair [31] — i.e. $\bar{J}$, $\bar{M}$ and $\bar{J} + \bar{M}$ are all Hamiltonian. The hierarchy (3.26) of bi-integrable couplings therefore possesses a bi-Hamiltonian structure (e.g. see [31, 32]), and hence is Liouville integrable. Further, it follows that there are infinitely many commuting common symmetries and conserved functionals

$$[\bar{K}_m, \bar{K}_n] = 0\,, \quad m, n \geq 0 \tag{3.47}$$

and

$$\{\bar{\mathscr{H}}_m, \bar{\mathscr{H}}_n\}_{\bar{J}} = \{\bar{\mathscr{H}}_m, \bar{\mathscr{H}}_n\}_{\bar{M}} = 0\,, \quad m, n \geq 0\,. \tag{3.48}$$

4. Concluding Remarks

We presented a matrix loop algebra consisting of 3×3 block matrices to construct bi-integrable couplings, and successfully generated a hierarchy of nonlinear bi-integrable couplings for the Dirac soliton hierarchy. The hierarchy of bi-integrable couplings possesses a bi-Hamiltonian structure, and thus infinitely many commuting common symmetries and conservation laws. The matrix loop algebra serves as a starting point of the formulation of constructing bi-integrable couplings, and the generating procedure can be applied to the other soliton hierarchies.

There are other interesting questions on integrable couplings. A crucial task in generating integrable couplings is to compute semi-direct sums of matrix loop algebras, but the issue is how to do so generally. Hamiltonian structures arise naturally in perturbation systems [15,33–35], but some enlarged spectral matrices are not associated with any non-degenerate bilinear forms over the underlying matrix loop algebras required in the variational identities [36,37]. We are curious about criteria that can be used to generate Hamiltonian structures for integrable couplings — e.g. Hamiltonian structures for

$$u_t = K(u)\,, \qquad u_{1,t} = K'(u)[u_1]\,, \qquad u_{2,t} = K'(u)[u_2]\,.$$

Can other matrix loop algebras, for example those obtained from the Kronecker product [38,39], help in formulating Hamiltonian structures? The construction of groups of solutions for integrable couplings by symmetry constraints similar to the theory for the perturbation systems [40,41], or by Darboux transformations generated through moving frames [42], also remains an open question. The problem is also related to the representation of solutions for linear partial differential equations with variable coefficients via the matrix exponential — e.g. see [43] for the case of ordinary differential equations.

Moreover, integrable couplings can inherit various other integrable characteristics [44], such as Hirota bilinear forms — e.g. see [45]. Another interesting property is whether the linear superposition principle can apply on subspaces of solutions of integrable couplings, and if the closure property for such subspaces of exponential wave solutions should contain different soliton solutions [46,47]. In particular, it is of interest to see what kinds of subspaces of solutions the above intriguing bi-integrable coupling can possess, or what relations there could exist between the system and Bell type polynomials. In order to develop the theory of multi-component integrable systems (e.g. see [48–54]), another future task is to explore distinct integrable properties for multi-integrable couplings from other points of view.

Finally, we mention our future intention to explore soliton phenomena that integrable couplings possess, with diverse intriguing structures in solution sets to nonlinear differential equations corresponding to dark energy and dark matter. Local bi-Hamiltonian models in (2+1)-dimensional integrable theories have been shown to exist, through studying bi-integrable couplings [13,35]. Particularly important solitons include resonant cases [55] carried by models of bi-integrable couplings.

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