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An Efficient and Stable Spectral-Element Method for Acoustic Scattering by an Obstacle

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Abstract. A spectral-element method is developed to solve the scattering problem for time-harmonic sound waves due to an obstacle in an homogeneous compressible fluid. The method is based on a boundary perturbation technique coupled with an efficient spectral-element solver. Extensive numerical results are presented, in order to show the accuracy and stability of the method.

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Key words: Spectral method, acoustic scattering, boundary perturbation, Helmholtz equation.

1. Introduction

The scattering of acoustic and electromagnetic waves is important in a wide range of problems of scientific and technological interest, and various numerical techniques have been proposed to solve related problems — cf. the book [3] and two surveys [16, 23] for example. A particularly compelling class of methods is based on boundary perturbations [2], and can be traced back to the work of Rayleigh [15] and Rice [17]. However, such algorithms depend upon significant cancellations for convergence, so their numerical implementations are highly ill-conditioned when pursued to high order [11, 12]. A new boundary perturbation algorithm called the transformed field expansion (TFE) that does not rely on strong cancellations for convergence was therefore proposed, and proved to be not only accurate and stable but also robust at high order [11–13].

The main purpose of this article is to apply the well developed method in Ref. [13] to the scattering problem for time-harmonic acoustic waves due to an obstacle in an homogeneous compressible fluid. In particular, we construct a well-conditioned spectral-element solver for the Helmholtz equation, which repeatedly involves the TFE algorithm. This solver is highly efficient and accurate because it exploits the fact that the Dirichlet-to-Neumann operator (DNO) is global in physical space but local in frequency space, hence

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we can reduce the two-dimensional problem to a sequence of one-dimensional problems using a spectrally accurate approximation to the DNO. We demonstrate that this provides an efficient numerical solution, as the frequency of the scattered radiation is varied from low to moderate.

In Section 2 we formulate the governing equations for an electromagnetic field incident upon a periodic two–dimensional irregular grating. In Section 3, we present the Method of Transformed Field Expansions, where we make a change of variables followed by an expansion in terms of the boundary perturbation. We then construct our two-domain spectral-element method to solve the resulting two–point boundary value problem in Section 4 and present the consequent numerical results in Section 5, followed by concluding remarks in Section 6. Finally, some of the detailed formulae needed in Section 3 are summarised in the Appendix.

2. Governing Equations

Given an incident field ϕ on a bounded obstacle $\Omega^+ \subset \mathbb{R}^2$ in the homogeneous compressible fluid, the scattered field in the exterior of the obstacle, $\Omega^- = \mathbb{R}^2 \backslash \bar{\Omega}^+$ can be written as

$$u^- = P_e - \phi \, ,$$

where P_e is the (total) acoustic pressure. The governing equation for u^- is

$$\Delta u^- + (k^-)^2 u^- = 0 \quad \text{in } \Omega^- \, , \tag{2.1}$$

where $k^- = \omega / c^-$ is the wave number (assumed to be real and positive), ω is the frequency, and c^- is the sound speed in Ω^-. We assume that the incident field ϕ satisfies (2.1) everywhere, except possibly at some places in Ω^- such that ϕ could correspond to a point source in Ω^- for example. We require that u^- satisfies the Sommerfeld radiation condition at infinity. Within the obstacle Ω^+, the reflected acoustic pressure u^+ satisfies the Bergmann equation [1,6]

$$\Delta u^+ + (k^+)^2 u^+ = 0, \quad \text{in } \Omega^+ \, , \tag{2.2}$$

where $k^+ = \omega / c^+$ with c^+ denoting the sound speed in Ω^+. At the interface $S = \partial \Omega^+$, we have a pair of transmission conditions expressing continuity of pressure and normal velocity — viz.

$$P_e = u^+ \;\text{ and }\; \frac{1}{\rho_-} \frac{\partial P_e}{\partial n} = \frac{1}{\rho_+} \frac{\partial u^+}{\partial n} \quad \text{on } S \, , \tag{2.3}$$