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A New High Accuracy Off-Step Discretisation for the Solution of 2D Nonlinear Triharmonic Equations

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Abstract. In this article, we derive a new fourth-order finite difference formula based on off-step discretisation for the solution of two-dimensional nonlinear triharmonic partial differential equations on a 9-point compact stencil, where the values of $u, (\partial^2 u/\partial n^2)$ and $(\partial^4 u/\partial n^4)$ are prescribed on the boundary. We introduce new ways to handle the boundary conditions, so there is no need to discretise the boundary conditions involving the partial derivatives. The Laplacian and biharmonic of the solution are obtained as a by-product of our approach, and we only need to solve a system of three equations. The new method is directly applicable to singular problems, and we do not require any fictitious points for computation. We compare its advantages and implementation with existing basic iterative methods, and numerical examples are considered to verify its fourth-order convergence rate.

AMS subject classifications: 65N06

Key words: High accuracy finite differences, off-step discretisation, two-dimensional nonlinear triharmonic equations, Laplacian, biharmonic, triharmonic, maximum absolute errors.

1. Introduction

We consider the numerical solution of the two-dimensional (2D) nonlinear triharmonic equation of the form

$$\varepsilon\nabla^6 u(x,y) \equiv \varepsilon\left(\frac{\partial^6 u}{\partial x^6} + 3\frac{\partial^6 u}{\partial x^4 \partial y^2} + 3\frac{\partial^6 u}{\partial x^2 \partial y^4} + \frac{\partial^6 u}{\partial y^6}\right)$$
$$= f(x,y,u,u_x,u_y,\nabla^2 u,\nabla^2 u_x,\nabla^2 u_y,\nabla^4 u,\nabla^4 u_x,\nabla^4 u_y)\,, \quad 0<x,y<1\,, \qquad (1.1)$$

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where $0 < \varepsilon \le 1$, $(x,y) \in \Omega = \{(x,y)|0 < x, y < 1\}$ with boundary $\partial\Omega$, and $\nabla^2 u(x,y) \equiv \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$ and $\nabla^4 u(x,y) \equiv \partial^4 u/\partial x^4 + 2\partial^4 u/(\partial x^2 \partial y^2) + \partial^4 u/\partial y^4$ represent the 2D Laplacian and biharmonic of the function $u(x,y)$. We assume that the solution $u(x,y)$ is smooth enough to maintain the order and accuracy of the scheme as high as possible. Dirichlet boundary conditions of the second kind are considered, given by

$$u = g_1(x,y), \quad \frac{\partial^2 u}{\partial n^2} = g_2(x,y), \quad \frac{\partial^4 u}{\partial n^4} = g_3(x,y), \quad (x,y) \in \partial\Omega. \tag{1.2}$$

The triharmonic equation (1.1) is a sixth-order elliptic partial differential equation encountered in viscous flow problems. Two-dimensional slowly rotating highly viscous flow in small cavities is modelled by the triharmonic equation for the stream function. However, few researchers have tried to solve triharmonic equations numerically, for it is difficult to discretise the differential equations and boundary conditions on a compact cell — and moreover, triharmonic problems require large computing power and a huge amount of memory that have begun to become available only recently.

Various techniques for the numerical solution of 2D nonlinear biharmonic equations have been considered in the literature, but not for 2D nonlinear triharmonic equations. A popular technique for the biharmonic equation is to split it into two coupled Poisson equations, each of which may be discretised using standard approximations and solved using a Poisson solver. A difficulty with this approach is that the boundary conditions for the new variable Laplacian introduced are not known and need to be approximated at the boundary. Smith [26] and Ehrlich [2, 3] have solved 2D biharmonic equations using coupled second-order accurate finite difference approximations, and Bauer and Riess [1] have used a block iterative method. Kwon *et al.* [7], Stephenson [28], Evans and Mohanty [4], and Mohanty *et al.* [9–12] subsequently developed certain second-order and fourth-order finite difference approximations for biharmonic problems using a 9-point compact cell. The compact cell approach involves discretising the biharmonic equations, using not just the grid values of the unknown solution u but also the values of the derivatives u_{xx}, u_{yy} and u_{zz} at the selected grid points. For 2D and 3D problems, these researchers solved systems of three and four equations to obtain the values of u, u_{xx}, u_{yy} and $u, u_{xx}, u_{yy}, u_{zz}$, respectively. Fourth-order compact finite difference schemes have become quite popular, compared with lower order schemes that require high mesh refinement and hence are less computationally efficient. The higher order accuracy of the fourth-order compact methods, combined with the compactness of the difference stencil, yields highly accurate numerical solutions on relatively coarse grids with greater computational efficiency.

One numerical approach for solving the 2D triharmonic equation (1.1) is to discretise the differential equation on a uniform grid using 49-point approximations with a truncation error of order h^2. This approximation connects central point values, in each case involving 48 neighbouring values of u in a 7×7 grid. The central value of u is connected to grid points three grids away in each direction from the central point, and the difference approximations need to be modified at grid points near the boundaries. However, in the solution of the linear and nonlinear systems obtained through such 49-point discretisation of the 2D triharmonic equation, there are serious computational difficulties that

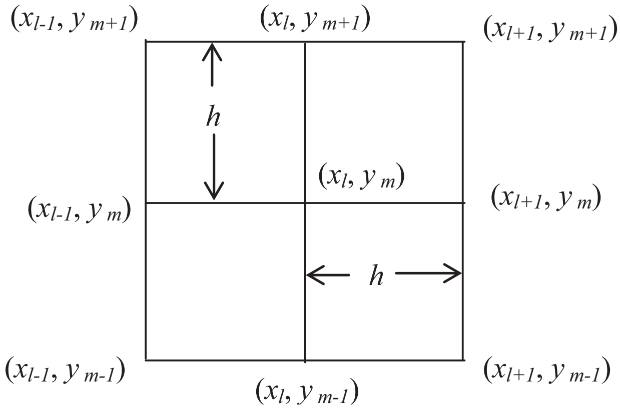

Figure 1: 9-point 2D single computational cell.

approximations using compact cells avoid. The compact cell approach previously involved discretising not only the grid values of the unknown solution u but also the values of the derivatives u_{xx} and u_{yy} at selected grid points [9]. Recently, Mohanty *et al.* [15–20] have developed single-cell compact finite difference discretisations of order two and four, for multi-dimensional biharmonic and triharmonic problems.

In this article, we split the differential equation (1.1) into a system of three Poisson equations, and introduce new ways to handle the boundary conditions that avoid discretising them in the system of equations. We require only a 9-point compact cell (cf. Fig. 1) and four off-step grid points, in a fourth-order approximation of the differential equation (1.1). The Dirichlet boundary conditions (1.2) are exactly satisfied, with no approximations required for the derivatives at the boundaries. The proposed new technique is not applicable to the triharmonic problem of the first kind, as we cannot obtain the 9-point compact cell fourth-order approximations in that case. However, the methods developed in our earlier work [19,20] were not directly applicable to singular problems without some modification, but the new method proposed here is.

In Section 2, we discuss the finite difference approximation for the differential equations (1.1), and in Section 3 we give a complete derivation of the method. In Section 4, we discuss block iterative methods, and in Section 5 we present stability analysis and illustrate the method and its fourth-order convergence by solving three problems. We compare the advantages and implementation of the proposed new method in the context of existing basic iterative methods. Our concluding remarks are made in Section 6.

2. Triharmonic Discretisation

Consider a 2D uniform grid centred at the point (x_l, y_m), where $h > 0$ is the constant mesh length in both the x and y directions, and $x_l = lh$, $y_m = mh$, $l, m = 0, 1, 2, \cdots, N$ with $(N+1)h = 1$. Let $U_{l,m}$ and $u_{l,m}$ be the exact and approximate solution values of $u(x, y)$ at the grid point (x_l, y_m), respectively. Recall that the Dirichlet boundary conditions are given by (1.2). Since the grid lines are parallel to coordinate axes and the values of u are exactly known on the boundary, this implies that the successive tangential partial derivatives of u are known exactly on the boundary. We follow the technique given by Mohanty [15].

The values of $u(x,0)$, $u_{yy}(x,0)$ and $u_{yyyy}(x,0)$ are known on the line $y = 0$, such that the values of $u_x(x,0)$, $u_{xx}(x,0)$, $u_{xxx}(x,0)$, $u_{xxxx}(x,0)$, $u_{yyx}(x,0)$, $u_{yyxx}(x,0)$, $\cdots$, etc. are also known there. This implies the values of $u(x,0)$, $\nabla^2 u(x,0) \equiv u_{xx}(x,0) + u_{yy}(x,0)$ and $\nabla^4 u(x,0) \equiv u_{xxxx}(x,0) + 2u_{xxyy}(x,0) + u_{yyyy}(x,0)$ are known on the line $y = 0$. Similarly, the values of u, $\nabla^2 u$ and $\nabla^4 u$ are known on all sides of the square region Ω. The Dirichlet boundary conditions (1.2) may be replaced by

$$u = g_1(x,y)\,, \quad \nabla^2 u = g_2(x,y)\,, \quad \nabla^4 u = g_3(x,y)\,, \quad (x,y) \in \partial\Omega\,. \tag{2.1}$$

Let us write $\nabla^2 u = v$ and $\nabla^2 v = w$. Then we can re-express the boundary value problem consisting of the partial differential equation (1.1) subject to the conditions (2.1) as a system of three Poisson equations of the form

$$\nabla^2 u(x,y) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = v(x,y)\,, \quad (x,y) \in \Omega\,, \tag{2.2a}$$

$$\nabla^2 v(x,y) \equiv \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = w(x,y), \quad (x,y) \in \Omega\,, \tag{2.2b}$$

$$\varepsilon \nabla^2 w(x,y) \equiv \varepsilon \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = f(x,y,u,v,w,u_x,v_x\,,w_x,u_y,v_y,w_y)\,, \quad (x,y) \in \Omega\,. \tag{2.2c}$$

and we have the exact Dirichlet boundary conditions for all three equations (2.2a)–(2.2c):

$$u = g_1(x,y)\,, \quad v = g_2(x,y)\,, \quad w = g_3(x,y)\,, \quad (x,y) \in \partial\Omega\,. \tag{2.3}$$

In passing, we note that for the first kind problem the values of $u(x,0)$, $u_x(x,0)$, $u_{xx}(x,0)$, $u_y(x,0), u_{xy}(x,0), u_{xxy}(x,0), \cdots$ etc. are known on the line $y = 0$, but we do not have any information about the values of $u_{yy}(x,0)$. Consequently, in that case we cannot evaluate $v(x,0)$ nor $w(x,0)$, and similarly we cannot find the value of $v(0,y)$ or $w(0,y)$ either, so the modified boundary value problem (2.2a)–(2.3) is inapplicable.

At the grid points (x_l, y_m), let us denote the exact and approximate solution values of $v(x,y)$ and $w(x,y)$ by $V_{l,m}, W_{l,m}$ and $v_{l,m}, w_{l,m}$, respectively. Fourth-order nine-point compact finite-difference methods for Poisson and harmonic equations are discussed by Jain [21], Collatz [22] and Ames [23]. For fourth-order approximation of the nonlinear differential equation (1.1) on the 9-point compact cell, we need the following approximations:

$$\bar{U}_{l\pm\frac{1}{2},m} = \frac{1}{2}\left(U_{l\pm1,m} + U_{l,m}\right), \tag{2.4a}$$

$$\bar{V}_{l\pm\frac{1}{2},m} = \frac{1}{2}\left(V_{l\pm1,m} + V_{l,m}\right), \tag{2.4b}$$

$$\bar{W}_{l\pm\frac{1}{2},m} = \frac{1}{2}\left(W_{l\pm1,m} + W_{l,m}\right), \tag{2.4c}$$

$$\bar{U}_{l,m\pm\frac{1}{2}} = \frac{1}{2}\left(U_{l,m\pm1} + U_{l,m}\right), \tag{2.5a}$$

$$\bar{V}_{l,m\pm\frac{1}{2}} = \frac{1}{2}\left(V_{l,m\pm1} + V_{l,m}\right), \tag{2.5b}$$

$$\bar{W}_{l,m\pm\frac{1}{2}} = \frac{1}{2}\left(W_{l,m\pm1} + W_{l,m}\right), \tag{2.5c}$$

$$\bar{U}_{xl,m} = \frac{1}{2h}\left(U_{l+1,m} - U_{l-1,m}\right), \tag{2.6a}$$

$$\bar{V}_{xl,m} = \frac{1}{2h}\left(V_{l+1,m} - V_{l-1,m}\right), \tag{2.6b}$$

$$\bar{W}_{xl,m} = \frac{1}{2h}\left(W_{l+1,m} - W_{l-1,m}\right), \tag{2.6c}$$

$$\bar{U}_{xl\pm\frac{1}{2},m} = \pm\frac{1}{h}\left(U_{l+1,m} \pm U_{l,m}\right), \tag{2.7a}$$

$$\bar{V}_{xl\pm\frac{1}{2},m} = \pm\frac{1}{h}\left(V_{l+1,m} \pm V_{l,m}\right), \tag{2.7b}$$

$$\bar{W}_{xl\pm\frac{1}{2},m} = \pm\frac{1}{h}\left(W_{l+1,m} \pm W_{l,m}\right), \tag{2.7c}$$

$$\bar{U}_{xl,m\pm\frac{1}{2}} = \frac{1}{4h}\left(U_{l+1,m\pm1} - U_{l-1,m\pm1} + U_{l+1,m} - U_{l-1,m}\right), \tag{2.8a}$$

$$\bar{V}_{xl,m\pm\frac{1}{2}} = \frac{1}{4h}\left(V_{l+1,m\pm1} - V_{l-1,m\pm1} + V_{l+1,m} - V_{l-1,m}\right), \tag{2.8b}$$

$$\bar{W}_{xl,m\pm\frac{1}{2}} = \frac{1}{4h}\left(W_{l+1,m\pm1} - W_{l-1,m\pm1} + W_{l+1,m} - W_{l-1,m}\right), \tag{2.8c}$$

$$\bar{U}_{yl,m} = \frac{1}{2h}\left(U_{l,m+1} - U_{l,m-1}\right), \tag{2.9a}$$

$$\bar{V}_{yl,m} = \frac{1}{2h}\left(V_{l,m+1} - V_{l,m-1}\right), \tag{2.9b}$$

$$\bar{W}_{yl,m} = \frac{1}{2h}\left(W_{l,m+1} - W_{l,m-1}\right), \tag{2.9c}$$

$$\bar{U}_{yl\pm\frac{1}{2},m} = \frac{1}{4h}\left(U_{l\pm1,m+1} - U_{l\pm1,m-1} + U_{l,m+1} - U_{l,m-1}\right), \tag{2.10a}$$

$$\bar{V}_{yl\pm\frac{1}{2},m} = \frac{1}{4h}\left(V_{l\pm1,m+1} - V_{l\pm1,m-1} + V_{l,m+1} - V_{l,m-1}\right), \tag{2.10b}$$

$$\bar{W}_{yl\pm\frac{1}{2},m} = \frac{1}{4h}\left(W_{l\pm1,m+1} - W_{l\pm1,m-1} + W_{l,m+1} - W_{l,m-1}\right), \tag{2.10c}$$

$$\bar{U}_{yl,m\pm\frac{1}{2}} = \pm\frac{1}{h}\left(U_{l,m\pm1} - U_{l,m}\right), \tag{2.11a}$$

$$\bar{V}_{yl,m\pm\frac{1}{2}} = \pm\frac{1}{h}\left(V_{l,m\pm1} - V_{l,m}\right), \tag{2.11b}$$

$$\bar{W}_{yl,m\pm\frac{1}{2}} = \pm\frac{1}{h}\left(W_{l,m\pm1} - W_{l,m}\right). \tag{2.11c}$$

Then we evaluate

$$\begin{aligned} \overline{F}_{l\pm\frac{1}{2},m} = f\Big(& x_{l\pm\frac{1}{2}}, y_m, U_{l\pm\frac{1}{2},m}, V_{l\pm\frac{1}{2},m}, W_{l\pm\frac{1}{2},m}, \overline{U}_{xl\pm\frac{1}{2},m}, \overline{V}_{xl\pm\frac{1}{2},m}, \bar{W}_{xl\pm\frac{1}{2},m}, \\ & \overline{U}_{yl\pm\frac{1}{2},m}, \overline{V}_{yl\pm\frac{1}{2},m}, \bar{W}_{yl\pm\frac{1}{2},m}\Big), \end{aligned} \tag{2.12}$$

$$\begin{aligned} \overline{F}_{l,m\pm\frac{1}{2}} = f\Big(& x_l, y_{m\pm\frac{1}{2}}, U_{l,m\pm\frac{1}{2}}, V_{l,m\pm\frac{1}{2}}, W_{l,m\pm\frac{1}{2}}, \overline{U}_{xl,m\pm\frac{1}{2}}, \overline{V}_{xl,m\pm\frac{1}{2}}, \bar{W}_{xl,m\pm\frac{1}{2}}, \\ & \overline{U}_{yl,m\pm\frac{1}{2}}, \overline{V}_{yl,m\pm\frac{1}{2}}, \bar{W}_{yl,m\pm\frac{1}{2}}\Big). \end{aligned} \tag{2.13}$$

Further, we define

$$\hat{U}_{l,m} = U_{l,m} + \frac{h^2}{4}V_{l,m}, \tag{2.14a}$$

$$\hat{V}_{l,m} = V_{l,m} + \frac{h^2}{4}W_{l,m}, \tag{2.14b}$$

$$\hat{W}_{l,m} = W_{l,m} + \frac{h^2}{4\varepsilon}\bar{F}_{l,m}, \tag{2.14c}$$

$$\hat{U}_{xl,m} = \bar{U}_{xl,m} + \frac{h}{8}\left(V_{l+1,m} - V_{l-1,m}\right), \tag{2.15a}$$

$$\hat{V}_{xl,m} = \bar{V}_{xl,m} + \frac{h}{8}\left(W_{l+1,m} - W_{l-1,m}\right), \tag{2.15b}$$

$$\hat{W}_{xl,m} = \bar{W}_{xl,m} + \frac{h}{4\varepsilon}\left(\bar{F}_{l+\frac{1}{2},m} - \bar{F}_{l-\frac{1}{2},m}\right), \tag{2.15c}$$

$$\hat{U}_{yl,m} = \bar{U}_{yl,m} + \frac{h}{8}\left(V_{l,m+1} - V_{l,m-1}\right), \tag{2.16a}$$

$$\hat{V}_{yl,m} = \bar{V}_{yl,m} + \frac{h}{8}\left(W_{l,m+1} - W_{l,m-1}\right), \tag{2.16b}$$

$$\hat{W}_{yl,m} = \bar{W}_{yl,m} + \frac{h}{4\varepsilon}\left(\bar{F}_{l,m+\frac{1}{2}} - \bar{F}_{l,m-\frac{1}{2}}\right). \tag{2.16c}$$

Finally, let

$$\hat{F}_{l,m} = f(x_l, y_m, \hat{U}_{l,m}, \hat{V}_{l,m}, \hat{W}_{l,m}, \hat{U}_{xl,m}, \hat{V}_{xl,m}, \hat{W}_{xl,m}, \hat{U}_{yl,m}, \hat{V}_{yl,m}, \hat{W}_{yl,m}). \tag{2.17}$$

Then at each internal grid point (x_l, y_m) of the solution region Ω, the given system of differential equations (2.2a)–(2.2c) is discretised by

$$\begin{aligned}
L[U] \equiv\ & U_{l-1,m-1}+4U_{l,m-1}+U_{l+1,m-1}+4U_{l-1,m}-20U_{l,m}+4U_{l+1,m}+U_{l-1,m+1}\\
&+4U_{l,m+1}+U_{l+1,m+1}\\
=\ & \frac{h^2}{2}\left[V_{l+1,m}+V_{l-1,m}+V_{l,m+1}+V_{l,m-1}+8V_{l,m}\right]+\mathrm{O}(h^6)\,,\\
& \qquad l,m=1(1)N\,, \qquad (2.18a)\\
L[V] \equiv\ & V_{l-1,m-1}+4V_{l,m-1}+V_{l+1,m-1}+4V_{l-1,m}-20V_{l,m}+4V_{l+1,m}+V_{l-1,m+1}\\
&+4V_{l,m+1}+V_{l+1,m+1}\\
=\ & \frac{h^2}{2}\left[W_{l+1,m}+W_{l-1,m}+W_{l,m+1}+W_{l,m-1}+8W_{l,m}\right]+\mathrm{O}(h^6)\,,\\
& \qquad l,m=1(1)N\,, \qquad (2.18b)\\
L[W] \equiv\ & \varepsilon\left[W_{l-1,m-1}+W_{l,m-1}+W_{l+1,m-1}+4W_{l-1,m}-20W_{l,m}+4W_{l+1,m}\right.\\
&\left.+W_{l-1,m+1}+4W_{l,m+1}+W_{l+1,m+1}\right]\\
=\ & 2h^2\left[\overline{F}_{l+\frac{1}{2},m}+\overline{F}_{l-\frac{1}{2},m}+\overline{F}_{l,m+\frac{1}{2}}+\overline{F}_{l,m-\frac{1}{2}}-\hat{F}_{l,m}\right]+\mathrm{O}(h^6)\,,\\
& \qquad l,m=1(1)N\,, \qquad (2.18c)
\end{aligned}$$

where the respective truncation errors are all $\mathrm{O}(h^6)$ as shown.

3. Derivation of the Numerical Method

To derive the new method, we follow Mohanty & Singh [13, 14]. At the grid point (x_l, y_m), we denote

$$\begin{aligned}
U_{ij} &= \frac{\partial^{i+j}U}{\partial {x_l}^i \partial {y_m}^j}\,, & V_{ij} &= \frac{\partial^{i+j}V}{\partial {x_l}^i \partial {y_m}^j}\,, & W_{ij} &= \frac{\partial^{i+j}W}{\partial {x_l}^i \partial {y_m}^j}\,,\\
\alpha^{(1)}_{l,m} &= \frac{\partial f}{\partial U_{l,m}}\,, & \alpha^{(2)}_{l,m} &= \frac{\partial f}{\partial V_{l,m}}\,, & \alpha^{(3)}_{l,m} &= \frac{\partial f}{\partial W_{l,m}}\,,\\
\beta^{(1)}_{l,m} &= \frac{\partial f}{\partial U_{xl,m}}\,, & \beta^{(2)}_{l,m} &= \frac{\partial f}{\partial V_{xl,m}}\,, & \beta^{(3)}_{l,m} &= \frac{\partial f}{\partial W_{xl,m}}\,,\\
\gamma^{(1)}_{l,m} &= \frac{\partial f}{\partial U_{yl,m}}\,, & \gamma^{(2)}_{l,m} &= \frac{\partial f}{\partial V_{yl,m}}\,, & \gamma^{(3)}_{l,m} &= \frac{\partial f}{\partial W_{yl,m}}\,, \qquad (3.1)
\end{aligned}$$

and at the grid point (x_l, y_m) define

$$F_{l,m} = f(x_l, y_m, U_{l,m}, V_{l,m}, W_{l,m}, U_{xl,m}, V_{xl,m}, W_{xl,m}, U_{yl,m}, V_{yl,m}, W_{yl,m})\,. \qquad (3.2)$$

Then adopting (3.1) and simplifying (2.4a)–(2.16c), we obtain

$$\overline{U}_{l\pm\frac{1}{2},m} = U_{l\pm\frac{1}{2},m} + \frac{h^2}{8}U_{20} + O(h^3)\,, \tag{3.3a}$$

$$\overline{V}_{l\pm\frac{1}{2},m} = V_{l\pm\frac{1}{2},m} + \frac{h^2}{8}V_{20} + O(h^3)\,, \tag{3.3b}$$

$$\overline{W}_{l\pm\frac{1}{2},m} = W_{l\pm\frac{1}{2},m} + \frac{h^2}{8}W_{20} + O(h^3)\,, \tag{3.3c}$$

$$\overline{U}_{l,m\pm\frac{1}{2}} = U_{l,m\pm\frac{1}{2}} + \frac{h^2}{8}U_{02} + O(h^3)\,, \tag{3.4a}$$

$$\overline{V}_{l,m\pm\frac{1}{2}} = V_{l,m+\frac{1}{2}} + \frac{h^2}{8}V_{02} + O(h^3)\,, \tag{3.4b}$$

$$\overline{W}_{l,m\pm\frac{1}{2}} = W_{l,m+\frac{1}{2}} + \frac{h^2}{8}W_{02} + O(h^3)\,, \tag{3.4c}$$

$$\overline{U}_{xl,m} = U_{xl,m} + \frac{h^2}{6}U_{30} + O(h^4)\,, \tag{3.5a}$$

$$\overline{V}_{xl,m} = V_{xl,m} + \frac{h^2}{6}V_{30} + O(h^4)\,, \tag{3.5b}$$

$$\overline{W}_{xl,m} = W_{xl,m} + \frac{h^2}{6}W_{30} + O(h^4)\,, \tag{3.5c}$$

$$\overline{U}_{xl\pm\frac{1}{2},m} = U_{xl\pm\frac{1}{2},m} + \frac{h^2}{24}U_{30} + O(h^4)\,, \tag{3.6a}$$

$$\overline{V}_{xl\pm\frac{1}{2},m} = V_{xl\pm\frac{1}{2},m} + \frac{h^2}{24}V_{30} + O(h^4)\,, \tag{3.6b}$$

$$\overline{W}_{xl\pm\frac{1}{2},m} = W_{xl\pm\frac{1}{2},m} + \frac{h^2}{24}W_{30} + O(h^4)\,, \tag{3.6c}$$

$$\overline{U}_{xl,m\pm\frac{1}{2}} = U_{xl,m\pm\frac{1}{2}} + \frac{h^2}{24}\left(4U_{30} + 3U_{12}\right) + O(h^3)\,, \tag{3.7a}$$

$$\overline{V}_{xl,m\pm\frac{1}{2}} = V_{xl,m\pm\frac{1}{2}} + \frac{h^2}{24}\left(4V_{30} + 3V_{12}\right) + O(h^3)\,, \tag{3.7b}$$

$$\overline{W}_{xl,m\pm\frac{1}{2}} = W_{xl,m\pm\frac{1}{2}} + \frac{h^2}{24}\left(4W_{30} + 3W_{12}\right) + O(h^3)\,, \tag{3.7c}$$

$$\overline{U}_{yl,m} = U_{yl,m} + \frac{h^2}{6}U_{03} + O(h^4)\,, \tag{3.8a}$$

$$\overline{V}_{yl,m} = V_{yl,m} + \frac{h^2}{6}V_{03} + O(h^4)\,, \tag{3.8b}$$

$$\overline{W}_{yl,m} = W_{yl,m} + \frac{h^2}{6}W_{03} + O(h^4)\,, \tag{3.8c}$$

$$\overline{U}_{yl\pm\frac{1}{2},m} = U_{yl\pm\frac{1}{2},m} + \frac{h^2}{24}\left(3U_{21} + 4U_{03}\right) + O(h^3)\,, \tag{3.9a}$$

$$\overline{V}_{yl\pm\frac{1}{2},m} = V_{yl\pm\frac{1}{2},m} + \frac{h^2}{24}\left(3V_{21} + 4V_{03}\right) + O(h^3), \tag{3.9b}$$

$$\overline{W}_{yl\pm\frac{1}{2},m} = W_{yl\pm\frac{1}{2},m} + \frac{h^2}{24}\left(3W_{21} + 4W_{03}\right) + O(h^3), \tag{3.9c}$$

$$\overline{U}_{yl,m\pm\frac{1}{2}} = U_{yl,m\pm\frac{1}{2}} + \frac{h^2}{24}U_{03} + O(h^3), \tag{3.10a}$$

$$\overline{V}_{yl,m\pm\frac{1}{2}} = V_{yl,m\pm\frac{1}{2}} + \frac{h^2}{24}V_{03} + O(h^3), \tag{3.10b}$$

$$\overline{W}_{yl,m\pm\frac{1}{2}} = W_{yl,m\pm\frac{1}{2}} + \frac{h^2}{24}W_{03} + O(h^3). \tag{3.10c}$$

At the grid point (x_l, y_m), we may write the difference equation (2.2c) as

$$\begin{aligned}&\varepsilon\left(\frac{\partial^2 W_{l,m}}{\partial x^2} + \frac{\partial^2 W_{l,m}}{\partial y^2}\right)\\ &= f(x, y, U_{l,m}, V_{l,m}, W_{l,m}, U_{xl,m}, V_{xl,m}, W_{xl,m}, U_{yl,m}, V_{yl,m}, W_{yl,m}) \equiv F_{l,m}.\end{aligned} \tag{3.11}$$

Using a Taylor expansion, we first obtain

$$\begin{aligned}&\varepsilon\left[\delta_x{}^2 + \delta_y{}^2 + \frac{1}{6}\delta_x{}^2\delta_y{}^2\right]W_{l,m}\\ =&\frac{h^2}{3}\left[F_{l+\frac{1}{2},m} + F_{l-\frac{1}{2},m} + F_{l,m+\frac{1}{2}} + F_{l,m-\frac{1}{2}} - F_{l,m}\right] + O\left(h^6\right),\end{aligned} \tag{3.12}$$

From the approximations (3.3a)–(3.10c), from (2.12)–(2.13), we have

$$\overline{F}_{l\pm\frac{1}{2},m} = F_{l\pm\frac{1}{2},m} + \frac{h^2}{24}T_1 \pm O\left(h^3\right), \tag{3.13a}$$

$$\overline{F}_{l,m\pm\frac{1}{2}} = F_{l,m\pm\frac{1}{2}} + \frac{h^2}{24}T_2 \pm O\left(h^3\right), \tag{3.13b}$$

where

$$\begin{aligned}T_1 =&3U_{20}\alpha^{(1)}_{l,m} + 3V_{20}\alpha^{(2)}_{l,m} + 3W_{20}\alpha^{(3)}_{l,m} + U_{30}\beta^{(1)}_{l,m} + V_{30}\beta^{(2)}_{l,m} + W_{30}\beta^{(3)}_{l,m}\\ &+ \left(3U_{21} + 4U_{03}\right)\gamma^{(1)}_{l,m} + \left(3V_{21} + 4V_{03}\right)\gamma^{(2)}_{l,m} + \left(3W_{21} + 4W_{03}\right)\gamma^{(3)}_{l,m},\\ T_2 =&3U_{02}\alpha^{(1)}_{l,m} + 3V_{02}\alpha^{(2)}_{l,m} + 3W_{02}\alpha^{(3)}_{l,m} + \left(3U_{12} + 4U_{30}\right)\beta^{(1)}_{l,m} + \left(3V_{12} + 4V_{30}\right)\beta^{(2)}_{l,m}\\ &+ \left(3W_{12} + 4W_{30}\right)\beta^{(3)}_{l,m} + U_{03}\gamma^{(1)}_{l,m} + V_{03}\gamma^{(2)}_{l,m} + W_{03}\gamma^{(3)}_{l,m}.\end{aligned}$$

Let

$$\hat{U}_{l,m} = U_{l,m} + a_1h^2V_{l,m}, \tag{3.14a}$$

$$\hat{V}_{l,m} = V_{l,m} + a_2h^2W_{l,m}, \tag{3.14b}$$

$$\hat{W}_{l,m} = W_{l,m} + a_3h^2\bar{F}_{l,m}, \tag{3.14c}$$

$$\hat{U}_{xl,m} = \overline{U}_{xl,m} + b_1 h \left(V_{l+1,m} - V_{l-1,m} \right) , \tag{3.15a}$$

$$\hat{V}_{xl,m} = \overline{V}_{xl,m} + b_2 h \left(W_{l+1,m} - W_{l-1,m} \right) , \tag{3.15b}$$

$$\hat{W}_{xl,m} = \overline{W}_{xl,m} + b_3 h \left(\overline{F}_{l+\frac{1}{2},m} - \overline{F}_{l-\frac{1}{2},m} \right) , \tag{3.15c}$$

$$\hat{U}_{yl,m} = \overline{U}_{yl,m} + c_1 h \left(V_{l,m+1} - V_{l,m-1} \right) , \tag{3.16a}$$

$$\hat{V}_{yl,m} = \overline{V}_{yl,m} + c_2 h \left(W_{l,m+1} - W_{l,m-1} \right) , \tag{3.16b}$$

$$\hat{W}_{yl,m} = \overline{W}_{yl,m} + c_3 h \left(\overline{F}_{l,m+\frac{1}{2}} - \overline{F}_{l,m-\frac{1}{2}} \right) , \tag{3.16c}$$

where a_1, a_2, a_3, b_1, b_2 , b_3, c_1, c_2, c_3 are parameters to be determined. Then with the help of the approximations (3.13a)–(3.13b) and simplifying (3.14a)–(3.16c), we obtain

$$\hat{U}_{l,m} = U_{l,m} + \frac{h^2}{6} T_3 + O\left(h^4\right) , \tag{3.17a}$$

$$\hat{V}_{l,m} = V_{l,m} + \frac{h^2}{6} T_3{}' + O\left(h^4\right) , \tag{3.17b}$$

$$\hat{W}_{l,m} = W_{l,m} + \frac{h^2}{6} T_3{}'' + O\left(h^4\right) , \tag{3.17c}$$

$$\hat{U}_{xl,m} = U_{xl,m} + \frac{h^2}{6} T_4 + O\left(h^4\right) , \tag{3.18a}$$

$$\hat{V}_{xl,m} = V_{xl,m} + \frac{h^2}{6} T_4{}' + O\left(h^4\right) , \tag{3.18b}$$

$$\hat{W}_{xl,m} = W_{xl,m} + \frac{h^2}{6} T_4{}'' + O\left(h^4\right) , \tag{3.18c}$$

$$\hat{U}_{yl,m} = U_{yl,m} + \frac{h^2}{6} T_5 + O\left(h^4\right) , \tag{3.19a}$$

$$\hat{V}_{yl,m} = V_{yl,m} + \frac{h^2}{6} T_5{}' + O\left(h^4\right) , \tag{3.19b}$$

$$\hat{W}_{yl,m} = W_{yl,m} + \frac{h^2}{6} T_5{}'' + O\left(h^4\right) , \tag{3.19c}$$

where

$$\begin{aligned}
&T_3 = 6a_1 \left(U_{20} + U_{02}\right), \\
&T_3{}' = 6a_2 \left(V_{20} + V_{02}\right), \\
&T_3{}'' = 6\varepsilon a_3 \left(W_{20} + W_{02}\right), \\
&T_4 = U_{30} + 12b_1 \left(U_{30} + U_{12}\right) = \left(1 + 12b_1\right) U_{30} + 12b_1 U_{12}, \\
&T_4{}' = V_{30} + 6b_2 \left(V_{30} + V_{12}\right) = \left(1 + 6b_2\right) V_{30} + 6b_2 V_{12}, \\
&T_4{}'' = W_{30} + 6\varepsilon b_3 \left(W_{30} + W_{12}\right) = \left(1 + 6\varepsilon b_3\right) W_{30} + 6\varepsilon b_3 W_{12},
\end{aligned}$$

$$T_5 = U_{03} + 6c_1\left(U_{03} + U_{21}\right) = \left(1 + 12c_1\right)U_{03} + 12c_1U_{21}\,,$$
$$T_5{}' = V_{03} + 6c_2\left(V_{03} + V_{21}\right) = \left(1 + 6c_2\right)V_{03} + 6c_2V_{21}\,,$$
$$T_5{}'' = W_{03} + 6\varepsilon c_3\left(W_{03} + W_{21}\right) = \left(1 + 6\varepsilon c_3\right)W_{03} + 6\varepsilon c_3W_{21}\,.$$

Now

$$\hat{F}_{l,m} = F_{l,m} + \frac{h^2}{6}T_6 + O(h^4)\,, \tag{3.20}$$

where

$$\begin{aligned} T_6 =& T_3\alpha^{(1)}_{l,m} + T'{}_3\alpha^{(2)}_{l,m} + T''{}_3\alpha^{(3)}_{l,m} + T_4\beta^{(1)}_{l,m} + T'{}_4\beta^{(2)}_{l,m} + T''{}_4\beta^{(3)}_{l,m} + T_5\gamma^{(1)}_{l,m} \\ &+ T'{}_5\gamma^{(2)}_{l,m} + T''{}_5\gamma^{(3)}_{l,m}. \end{aligned}$$

Substituting the approximations (3.13a)–(3.13b) and (3.20) into (2.18c) and noting (3.12), we obtain the local truncation error

$$\bar{T}_{l,m} = -\frac{h^2}{6}\left[T_1 + T_2 - 2T_6\right] + O(h^6)\,. \tag{3.21}$$

For the proposed new difference method to be fourth-order, the coefficient of h^4 in (3.21) must be zero, such that

$$T_1 + T_2 - 2T_6 = 0\,. \tag{3.22}$$

Then substituting the values of T_1, T_2 and T_6 in (3.22), we obtain the parameter values

$$\begin{aligned} a_1 &= \frac{1}{4}\,, & a_2 &= \frac{1}{4}\,, & a_3 &= \frac{1}{4\varepsilon}\,, \\ b_1 &= \frac{1}{8}\,, & b_2 &= \frac{1}{8}\,, & b_3 &= \frac{1}{4\varepsilon}\,, \\ c_1 &= \frac{1}{8}\,, & c_2 &= \frac{1}{8}\,, & c_3 &= \frac{1}{4\varepsilon}\,, \end{aligned}$$

and the local truncation error (3.21) reduces to $\overline{T}_{l,m} = O(h^6)$.

4. Block Iterative Methods

On combining the difference equations at each internal grid point, we obtain a large sparse matrix system to solve. At each interior mesh point, we have three unknowns u, $\nabla^2 u \equiv v$ and $\nabla^2 v \equiv w$ — i.e. the number of bands with non-zero entries is increased, and so is the size of the final matrix for the same mesh size. However, the values of the Laplacian and the biharmonic that are often of interest are also computed in this new method.

Whenever $f(x,y,u,v,w,u_x,v_x,w_x,u_y,v_y,w_y)$ is linear in u, v, w, u_x, v_x, w_x, u_y, v_y and w_y, the difference equations (2.18a)–(2.18c) form a linear block system. To solve such a system, or indeed to demonstrate the existence of a solution, one can use a block iterative method [5,6,8,24,25,29–31]. For a block iterative method, we first write (2.18a)–(2.18c) in the form

$$\boldsymbol{A}\boldsymbol{u}+\boldsymbol{B}\boldsymbol{v}+0=0\,, \tag{4.1a}$$

$$0+\boldsymbol{A}\boldsymbol{v}+\boldsymbol{B}\boldsymbol{w}=0\,, \tag{4.1b}$$

$$0+0+\boldsymbol{A}\boldsymbol{w}=\boldsymbol{c}\,, \tag{4.1c}$$

where $\boldsymbol{A}_{\rm L}=[1,4,1]$, $\boldsymbol{A}_{\rm D}=[4,-20,4]$, $\boldsymbol{A}_{\rm U}=[1,4,1]$ represent the lower, main and upper tridiagonal matrices of the tri-block diagonal matrix $\boldsymbol{A}=[\boldsymbol{A}_{\rm L},\boldsymbol{A}_{\rm D},\boldsymbol{A}_{\rm U}]$ and $\boldsymbol{B}_{\rm L}=[0,1,0]$, $\boldsymbol{B}_{\rm D}=[1,8,1]$, $\boldsymbol{B}_{\rm U}=[0,1,0]$ are the corresponding tridiagonal matrices of the tri-block diagonal matrix $\boldsymbol{B}=(-h^2/2)[\boldsymbol{B}_{\rm L},\boldsymbol{B}_{\rm D},\boldsymbol{B}_{\rm U}]$, $\{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\}$ is the set of solution vectors, and $\boldsymbol{c}$ is the vector consisting of the functions on the right-hand side and associated boundary conditions. Although the system (4.1a)–(4.1c) can be solved by various methods, block iterative methods work well. The block Gauss-Seidel (BGS) iterative method [24,25,29–31] may be written

$$\boldsymbol{A}_{\rm D}\boldsymbol{w}^{(k+1)}=-(\boldsymbol{A}_L+\boldsymbol{A}_{\rm U})\boldsymbol{w}^{(k)}+\boldsymbol{c}\,, \tag{4.2a}$$

$$\boldsymbol{A}_{\rm D}\boldsymbol{v}^{(k+1)}=-(\boldsymbol{A}_L+\boldsymbol{A}_{\rm U})\boldsymbol{v}^{(k)}-B\boldsymbol{w}^{(k+1)}\,, \tag{4.2b}$$

$$\boldsymbol{A}_{\rm D}\boldsymbol{u}^{(k+1)}=-(\boldsymbol{A}_L+\boldsymbol{A}_{\rm U})\boldsymbol{u}^{(k)}-B\boldsymbol{v}^{(k+1)}\,, \tag{4.2c}$$

and this system of equations can be solved using a tridiagonal solver.

Whenever $f(x,y,u,v,w,u_x,v_x,w_x,u_y,v_y,w_y)$ is nonlinear in u, v, w, u_x, v_x, w_x, u_y, v_y and w_y, the difference equations (2.18a)–(2.18c) form a nonlinear block system. To solve such a system, one can apply the Newton nonlinear block iterative method [5,6,8,24,25,29–31]. To define the nonlinear BGS method, we first write (2.18a)–(2.18c) in the form

$$\boldsymbol{A}\boldsymbol{u}+\boldsymbol{B}\boldsymbol{v}+0=0, \tag{4.3a}$$

$$0+\boldsymbol{A}\boldsymbol{v}+\boldsymbol{B}\boldsymbol{w}=0, \tag{4.3b}$$

$$H(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=0, \tag{4.3c}$$

where $\boldsymbol{A}=[\boldsymbol{A}_{\rm L},\boldsymbol{A}_{\rm D},\boldsymbol{A}_{\rm U}]$ and $\boldsymbol{B}=[\boldsymbol{B}_{\rm L},\boldsymbol{B}_{\rm D},\boldsymbol{B}_{\rm U}]$ are tri-block diagonal matrices defined earlier, and $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are solution vectors of the linear system (4.3a), (4.3b) and nonlinear system (4.3c). Now we compute the values of $\boldsymbol{u}$, $\boldsymbol{v}$ from (4.3a) and (4.3b) using a linear iterative method, and value of $\boldsymbol{w}$ from (4.3c) using a nonlinear iterative method. The Jacobian $\boldsymbol{J}$ of $\boldsymbol{H}$ is easily found to be the block tridiagonal matrix $\boldsymbol{J}=[\boldsymbol{J}_{\rm L},\boldsymbol{J}_{\rm D},\boldsymbol{J}_{\rm U}]$, where

$$\boldsymbol{J}_{\rm L}=\left[\frac{\partial H}{\partial w_{l-1,m-1}},\ \frac{\partial H}{\partial w_{l,m-1}},\ \frac{\partial H}{\partial w_{l+1,m-1}}\right],$$

$$\boldsymbol{J}_{\rm D}=\left[\frac{\partial H}{\partial w_{l-1,m}},\ \frac{\partial H}{\partial w_{l,m}},\ \frac{\partial H}{\partial w_{l+1,m}}\right],$$

and

$$J_{\mathrm{U}} = \left[\frac{\partial H}{\partial w_{l-1,m+1}}, \frac{\partial H}{\partial w_{l,m+1}}, \frac{\partial H}{\partial w_{l+1,m+1}} \right]$$

are N^{th} order tridiagonal matrices. The matrix equation for the Newton BGS method is then given by

$$\boldsymbol{J}\Delta \boldsymbol{w}^{(k)} = -H(\boldsymbol{u}^{(k+1)}, \boldsymbol{v}^{(k+1)}, \boldsymbol{w}^{(k)}), \tag{4.4}$$

where $(\boldsymbol{u}^{(0)}, \boldsymbol{v}^{(0)}, \boldsymbol{w}^{(0)})$ is the initial approximation of $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$, $\Delta \boldsymbol{w}^{(k)}$ is any intermediate vector and the values of $\boldsymbol{u}^{(k+1)}, \boldsymbol{v}^{(k+1)}$ are known from the previous step. We define

$$\boldsymbol{w}^{(k+1)} = \boldsymbol{w}^{(k)} + \Delta \boldsymbol{w}^{(k)}, \qquad k = 0, 1, 2, \cdots . \tag{4.5}$$

We can thus represent (4.3a)–(4.3c) as follows:

$$\boldsymbol{A}_{\mathrm{D}}\boldsymbol{u}^{(k+1)} = -(\boldsymbol{A}_{\mathrm{L}} + \boldsymbol{A}_{\mathrm{U}})\boldsymbol{u}^{(k)} - \boldsymbol{B}\boldsymbol{v}^{(k)}, \quad k = 0, 1, 2, \cdots , \tag{4.6a}$$

$$\boldsymbol{A}_{\mathrm{D}}\boldsymbol{v}^{(k+1)} = -(\boldsymbol{A}_{\mathrm{L}} + \boldsymbol{A}_{\mathrm{U}})\boldsymbol{v}^{(k)} - \boldsymbol{B}\boldsymbol{w}^{(k)}, \quad k = 0, 1, 2, \cdots , \tag{4.6b}$$

$$\boldsymbol{J}_{\mathrm{D}}\Delta \boldsymbol{w}^{(k+1)} = -H(\boldsymbol{u}^{(k+1)}, \boldsymbol{v}^{(k+1)}, \boldsymbol{w}^{(k)}) - (\boldsymbol{J}_{\mathrm{L}} + \boldsymbol{J}_{\mathrm{U}})\Delta \boldsymbol{w}^{(k)}, \quad k = 0, 1, 2, \cdots . \tag{4.6c}$$

This system can be solved by using a tridiagonal solver. By using the outer iterative method (4.5), we can then evaluate $\boldsymbol{w}^{(k+1)}$, $k = 0, 1, 2, \cdots$. In order for this method to converge, the initial iterate $(\boldsymbol{u}^{(0)}, \boldsymbol{v}^{(0)}, \boldsymbol{w}^{(0)})$ must be sufficiently close to the solution.

The second order approximations for the system of differential equations (2.2a)–(2.2c) are straightforward and can be written

$$U_{l,m-1} + U_{l-1,m} - 4U_{l,m} + U_{l+1,m} + U_{l,m+1} = h^2 V_{l,m} + O(h^4), \quad l, m = 1(1)N , \tag{4.7a}$$

$$V_{l,m-1} + V_{l-1,m} - 4V_{l,m} + V_{l+1,m} + V_{l,m+1} = h^2 W_{l,m} + O(h^4), \quad l, m = 1(1)N , \tag{4.7b}$$

$$\begin{aligned} &W_{l,m-1} + W_{l-1,m} - 4W_{l,m} + W_{l+1,m} + W_{l,m+1} \\ &\quad = h^2 f(x_l, y_m, U_{l,m}, V_{l,m}, W_{l,m}, \overline{U}_{xl,m}, \overline{V}_{xl,m}, \overline{W}_{xl,m}, \overline{U}_{yl,m}, \overline{V}_{yl,m}, \overline{W}_{yl,m}) + O(h^4), \\ &\qquad l, m = 1(1)N . \end{aligned} \tag{4.7c}$$

Note that these second order approximations (4.7a)–(4.7c) require only 5-grid points on a single computational cell (cf. Fig. 1), applicable to linear triharmonic problems with singular coefficients. Similarly, we can discuss the block iterative methods for the system (4.7a)–(4.7c).

5. Stability Analysis and Experimental Results

Let us consider the test equation

$$\nabla^6 u = g(x, y), \quad 0 < x, y < 1 . \tag{5.1}$$

Applying the proposed method (2.18a)–(2.18c), we obtain

$$\begin{aligned}&U_{l-1,m-1}+4U_{l,m-1}+U_{l+1,m-1}+4U_{l-1,m}-20U_{l,m}+4U_{l+1,m}+U_{l-1,m+1}+4U_{l,m+1}\\&\quad+U_{l+1,m+1}=\frac{h^2}{2}\left[V_{l+1,m}+V_{l-1,m}+V_{l,m+1}+V_{l,m-1}+8V_{l,m}\right],\\&\qquad l,m=1(1)N,\end{aligned}\tag{5.2a}$$

$$\begin{aligned}&V_{l-1,m-1}+4V_{l,m-1}+V_{l+1,m-1}+4V_{l-1,m}-20V_{l,m}+4V_{l+1,m}+V_{l-1,m+1}+4V_{l,m+1}\\&\quad+V_{l+1,m+1}=\frac{h^2}{2}\left[W_{l+1,m}+W_{l-1,m}+W_{l,m+1}+W_{l,m-1}+8W_{l,m}\right],\\&\qquad l,m=1(1)N,\end{aligned}\tag{5.2b}$$

$$\begin{aligned}&W_{l-1,m-1}+4W_{l,m-1}+W_{l+1,m-1}+4W_{l-1,m}-20W_{l,m}+4W_{l+1,m}+W_{l-1,m+1}+4W_{l,m+1}\\&\quad+W_{l+1,m+1}=2h^2\left[g_{l+\frac{1}{2},m}+g_{l-\frac{1}{2},m}+g_{l,m+\frac{1}{2}}+g_{l,m-\frac{1}{2}}-g_{l,m}\right],\\&\qquad l,m=1(1)N,\end{aligned}\tag{5.2c}$$

where $g_{l,m}=g(x_l,y_m)$, $g_{l\pm\frac{1}{2},m}=g(x_l\pm 1/2,y_m)$ etc.. An iterative method for (5.2a)–(5.2c) can be written as

$$20\boldsymbol{I}\boldsymbol{u}^{(k+1)}=\boldsymbol{A}\boldsymbol{u}^{(k)}-\frac{h^2}{2}\boldsymbol{B}\boldsymbol{v}^{(k)}+0\boldsymbol{w}^{(k)}+\boldsymbol{RHU},\tag{5.3a}$$

$$20\boldsymbol{I}\boldsymbol{v}^{(k+1)}=0\boldsymbol{u}^{(k)}+\boldsymbol{A}\boldsymbol{v}^{(k)}-\frac{h^2}{2}\boldsymbol{B}\boldsymbol{w}^{(k)}+\boldsymbol{RHV},\tag{5.3b}$$

$$20\boldsymbol{I}\boldsymbol{w}^{(k+1)}=0\boldsymbol{u}^{(k)}+0\boldsymbol{v}^{(k)}+\boldsymbol{A}\boldsymbol{w}^{(k)}+\boldsymbol{RHW},\tag{5.3c}$$

where $\boldsymbol{u}^{(k)}$, $\boldsymbol{v}^{(k)}$, $\boldsymbol{w}^{(k)}$ are solution vectors and $\boldsymbol{RHU}$, $\boldsymbol{RHV}$, $\boldsymbol{RHW}$ are right-hand side vectors consisting of boundary and homogenous function values. The system (5.3) can be rewritten in matrix form as

$$\begin{bmatrix}\boldsymbol{U}^{(k+1)}\\\boldsymbol{V}^{(k+1)}\\\boldsymbol{W}^{(k+1)}\end{bmatrix}=\boldsymbol{G}\begin{bmatrix}\boldsymbol{U}^{(k)}\\\boldsymbol{V}^{(k)}\\\boldsymbol{W}^{(k)}\end{bmatrix}+\boldsymbol{RH},\tag{5.4}$$

where

$$\boldsymbol{G}=\frac{1}{20}\begin{bmatrix}\boldsymbol{A}&\frac{-h^2}{2}\boldsymbol{B}&0\\0&\boldsymbol{A}&\frac{-h^2}{2}\boldsymbol{B}\\0&0&\boldsymbol{A}\end{bmatrix},\qquad \boldsymbol{RH}=\begin{bmatrix}\boldsymbol{RHU}\\\boldsymbol{RHV}\\\boldsymbol{RHW}\end{bmatrix},$$

$$\begin{aligned}&\boldsymbol{A}=[\boldsymbol{P},\boldsymbol{Q},\boldsymbol{P}],\quad \boldsymbol{B}=[\boldsymbol{T},\boldsymbol{S},\boldsymbol{T}],\quad \boldsymbol{P}=[1,4,1],\\&\boldsymbol{Q}=[4,0,4],\quad \boldsymbol{T}=[0,1,0],\quad \boldsymbol{S}=[1,8,1],\end{aligned}$$

and we denote

$$[a,b,c]=\begin{bmatrix} b & c & & & 0 \\ a & b & c & & \\ & & \ddots & & \\ & & a & b & c \\ 0 & & & a & b \end{bmatrix}_{N\times N}$$

as the Nth order tridiagonal matrix with eigenvalues given by

$$\lambda_j = b + 2\sqrt{ac}\cos\left(\frac{\pi j}{N+1}\right), \quad j = 1,2,\cdots,N .$$

The above iterative method is stable provided $\rho(\boldsymbol{G}) \le 1$, where $\rho(\boldsymbol{G})$ is the spectral radius of $\boldsymbol{G}$. The eigenvalues of $\boldsymbol{Q}$ are given by

$$\lambda_k = 8\cos\frac{k\pi}{N+1} \equiv 8\cos(k\pi h), \quad k = 1(1)N, \tag{5.5}$$

and the eigenvalues of $\boldsymbol{P}$ are given by

$$\mu_k = 4 + 2\cos\frac{k\pi}{N+1} \equiv 4 + 2\cos(k\pi h), \quad k = 1(1)N . \tag{5.6}$$

Consequently, the eigenvalues of $\boldsymbol{A}$ are given by

$$\begin{aligned} \nu_{jk} =& \lambda_k + 2\mu_k\cos(j\pi h) \equiv 8[\cos(k\pi h)+\cos(j\pi h)] + 4\cos(k\pi h)\cos(j\pi h), \\ & j = 1(1)N,\ k = 1(1)N, \end{aligned} \tag{5.7}$$

and the eigenvalues of $\boldsymbol{G}$ are

$$\begin{aligned} \xi_{jk} =& \frac{1}{20}\nu_{jk} = \frac{1}{20}[8(\cos(k\pi h)+\cos(j\pi h)) + 4\cos(k\pi h)\cos(j\pi h)], \\ & j = 1(1)N,\ k = 1(1)N . \end{aligned} \tag{5.8}$$

The maximum eigenvalue of $\boldsymbol{G}$ occurs at $j = k = 1$. Hence

$$\rho(\boldsymbol{G}) = \max\left|\xi_{jk}\right| = \frac{\cos(\pi h)}{5}[4+\cos(\pi h)] \le 1, \tag{5.9}$$

which is satisfied for all variable angles πh, so the iterative method (5.3a)–(5.3c) is stable.

In order to validate the proposed fourth-order method and test its robustness, in the region $0 < x, y < 1$ we solve the following three test problems with known exact solutions. The Dirichlet boundary conditions and right-hand side homogeneous functions are obtained from the exact solutions. We solved the linear systems using the block Gauss-Seidel iterative method, and the nonlinear system of equations by the Newton block Gauss-Seidel iterative method. We also compared the numerical results obtained by the proposed fourth-order approximations (2.18a)–(2.18c) with the numerical results obtained via the second-order approximations (4.7a)–(4.7c). In all cases, we considered $\boldsymbol{u}^{(0)} = 0$ as the initial approximation, and stopped the iterations when the absolute error tolerance $|\boldsymbol{u}^{(k+1)} - \boldsymbol{u}^{(k)}| \leq 10^{-12}$ was achieved. In all cases, we calculated maximum absolute errors (l_∞-norm) for different grid sizes, and all computation was performed in double precision arithmetic.

Example 5.1. (Test problem)

Two-dimensional triharmonic problem (5.1) in a unit square. The exact solution is $u(x,y) = \sin(\pi x)\cdot\sin(\pi y)$.

The maximum absolute errors are tabulated in Table 1.

Table 1: The maximum absolute errors for Example 5.1.

h	Proposed $O(h^4)$ - Method	$O(h^2)$ - Method
u	0.4487(−03)	0.3935(−01)
$1/8\nabla^2 u$	0.7567(−02)	0.5145(+00)
$\nabla^4 u$	0.1238(+00)	0.5046(+01)
u	0.2791(−04)	0.9688(−02)
$1/16\nabla^2 u$	0.4697(−03)	0.1272(+00)
$\nabla^4 u$	0.7666(−02)	0.1254(+01)
u	0.1742(−05)	0.2412(−02)
$1/32\nabla^2 u$	0.2930(−04)	0.3173(−01)
$\nabla^4 u$	0.4799(−03)	0.3131(+00)
u	0.1088(−06)	0.6025(−03)
$1/64\nabla^2 u$	0.1830(−05)	0.7928(−02)
$\nabla^4 u$	0.2985(−04)	0.7824(−01)

Example 5.2. (Singular Problem)

$$\nabla^6 u + \frac{1}{x}\left(\frac{\partial^5 u}{\partial x^5} + 2\frac{\partial^5 u}{\partial x^3 \partial y^2} + \frac{\partial^5 u}{\partial x \partial y^4}\right) = f(x,y), \qquad 0 < x, y < 1. \tag{5.10}$$

The exact solution is $u(x,y) = x^2 \sin(\pi y)$.

The maximum absolute errors are tabulated in Table 2.

Example 5.3. (Navier-Stokes model equation in terms of stream function ψ, see [26])

$$\begin{aligned}\frac{1}{R_e}\nabla^6\psi =& \psi_y(\nabla^2\psi)_x - \psi_x(\nabla^2\psi)_y + (\nabla^2\psi)_y(\nabla^4\psi)_x - (\nabla^2\psi)_x(\nabla^4\psi)_y \\ &+ \psi_x(\nabla^4\psi)_y - \psi_y(\nabla^4\psi)_x + G(x,y), \qquad 0 < x, y < 1. \end{aligned} \tag{5.11}$$

The exact solution is$\psi(x,y) = e^x \cos(\pi y)$.

The maximum absolute errors are tabulated in Table 3, for various values of the Reynolds number R_e.

Table 2: The maximum absolute errors for Example 5.2.

h	Proposed $O(h^4)$ - Method	$O(h^4)$ - Method discussed in [19]	$O(h^2)$ - Method
u	0.7265(−04)	0.8884(−04)	0.2858(−02)
$1/8\nabla^2 u$	0.6715(−03)	0.8118(−03)	0.1294(−01)
$\nabla^4 u$	0.8106(−02)	0.1121(−01)	0.1428(+00)
u	0.4616(−05)	0.6162(−05)	0.755(7 − 03)
$1/16\nabla^2 u$	0.4218(−04)	0.5316(−04)	0.3292(−02)
$\nabla^4 u$	0.5158(−03)	0.7963(−03)	0.3931(−01)
u	0.2892(−06)	0.4242(−06)	0.1891(−03)
$1/32\nabla^2 u$	0.2655(−05)	0.3818(−05)	0.8294(−03)
$\nabla^4 u$	0.3265(−04)	0.5810(−04)	0.1025(−01)
u	0.1808(−07)	0.2812(−07)	0.4731(−04)
$1/64\nabla^2 u$	0.1665(−06)	0.2522(−06)	0.2071(−03)
$\nabla^4 u$	0.2063(−05)	0.3836(−05)	0.2658(−02)

Table 3: The maximum absolute errors for Example 5.3.

h	Proposed $O(h^4)$ - Method		$O(h^4)$ - Method discussed in [19]		$O(h^2)$ - Method
	$R_e = 10^2$	$R_e = 10^4, 10^6, 10^8$	$R_e = 10^2$	$R_e = 10^4, 10^6, 10^8$	$R_e = 10^2, 10^4, 10^6, 10^8$
ψ	0.4760(−04)	0.4740(−04)	0.8255(−04)	0.8230(−04)	
$1/8\nabla^2\psi$	0.4302(−03)	0.4205(−03)	0.7834(−03)	0.7624(−03)	Over Flow
$\nabla^4\psi$	0.4212(−02)	0.3734(−02)	0.7664(−02)	0.7112(−02)	
ψ	0.3001(−05)	0.2952(−05)	0.5435(−05)	0.5216(−05)	
$1/16\nabla^2\psi$	0.2859(−04)	0.2620(−04)	0.4832(−04)	0.4544(−04)	Over Flow
$\nabla^4\psi$	0.3625(−03)	0.2334(−03)	0.5016(−03)	0.4228(−03)	
ψ	0.1972(−06)	0.1820(−06)	0.3226(−06)	0.3184(−06)	
$1/32\nabla^2\psi$	0.2149(−05)	0.1639(−05)	0.2819(−05)	0.2787(−05)	Over Flow
$\nabla^4\psi$	0.4316(−04)	0.1433(−04)	0.4006(−04)	0.2582(−04)	
ψ	0.1124(−07)	0.1055(−07)	0.2026(−07)	0.1892(−07)	
$1/64\nabla^2\psi$	0.1316(−06)	0.8378(−07)	0.1811(−06)	0.1774(−06)	Over Flow
$\nabla^4\psi$	0.3348(−05)	0.6102(−06)	0.2883(−05)	0.1665(−05)	

6. Conclusions

In this article, we developed a new fourth-order compact finite difference method based on off-step discretisation for the solution of 2D nonlinear triharmonic partial differential equations. The method involves a 9-point compact stencil with the values of u, the Laplacian and the biharmonic as unknowns. We obtain the Laplacian and biharmonic of u as by-products, which are quite often of interest in many applied mathematics problems. Our numerical experiments confirmed that the proposed fourth-order discretisation produces oscillation-free solutions for high Reynolds number, whereas a second order discretisation is unstable. We have compared the results obtained using the new method proposed here with the results obtained in Ref [19]. The results from the new method are slightly better, but its main advantages are that it is directly applicable irrespective of the coordinate system and we do not need to modify our method for singular problems. We are currently working to apply the new method to 3D nonlinear triharmonic elliptic and time-dependent parabolic partial differential equations.

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