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A New High Accuracy Off-Step Discretisation for the Solution of 2D Nonlinear Triharmonic Equations

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Abstract. In this article, we derive a new fourth-order finite difference formula based on off-step discretisation for the solution of two-dimensional nonlinear triharmonic partial differential equations on a 9-point compact stencil, where the values of $u, (\partial^2 u/\partial n^2)$ and $(\partial^4 u/\partial n^4)$ are prescribed on the boundary. We introduce new ways to handle the boundary conditions, so there is no need to discretise the boundary conditions involving the partial derivatives. The Laplacian and biharmonic of the solution are obtained as a by-product of our approach, and we only need to solve a system of three equations. The new method is directly applicable to singular problems, and we do not require any fictitious points for computation. We compare its advantages and implementation with existing basic iterative methods, and numerical examples are considered to verify its fourth-order convergence rate.

AMS subject classifications: 65N06

Key words: High accuracy finite differences, off-step discretisation, two-dimensional nonlinear triharmonic equations, Laplacian, biharmonic, triharmonic, maximum absolute errors.

1. Introduction

We consider the numerical solution of the two-dimensional (2D) nonlinear triharmonic equation of the form

$$\begin{aligned}\varepsilon\nabla^6 u(x,y) &\equiv \varepsilon\left(\frac{\partial^6 u}{\partial x^6}+3\frac{\partial^6 u}{\partial x^4\partial y^2}+3\frac{\partial^6 u}{\partial x^2\partial y^4}+\frac{\partial^6 u}{\partial y^6}\right)\\ &= f(x,y,u,u_x,u_y,\nabla^2 u,\nabla^2 u_x,\nabla^2 u_y,\nabla^4 u,\nabla^4 u_x,\nabla^4 u_y)\,, \quad 0<x,y<1\,, \qquad (1.1)\end{aligned}$$

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where $0 < \varepsilon \leq 1$, $(x, y) \in \Omega = \{(x, y)|0 < x, y < 1\}$ with boundary $\partial\Omega$, and $\nabla^2 u(x, y) \equiv \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$ and $\nabla^4 u(x, y) \equiv \partial^4 u/\partial x^4 + 2\partial^4 u/(\partial x^2 \partial y^2) + \partial^4 u/\partial y^4$ represent the 2D Laplacian and biharmonic of the function $u(x, y)$. We assume that the solution $u(x, y)$ is smooth enough to maintain the order and accuracy of the scheme as high as possible. Dirichlet boundary conditions of the second kind are considered, given by

$$u = g_1(x, y), \quad \frac{\partial^2 u}{\partial n^2} = g_2(x, y), \quad \frac{\partial^4 u}{\partial n^4} = g_3(x, y), \quad (x, y) \in \partial\Omega. \tag{1.2}$$

The triharmonic equation (1.1) is a sixth-order elliptic partial differential equation encountered in viscous flow problems. Two-dimensional slowly rotating highly viscous flow in small cavities is modelled by the triharmonic equation for the stream function. However, few researchers have tried to solve triharmonic equations numerically, for it is difficult to discretise the differential equations and boundary conditions on a compact cell — and moreover, triharmonic problems require large computing power and a huge amount of memory that have begun to become available only recently.

Various techniques for the numerical solution of 2D nonlinear biharmonic equations have been considered in the literature, but not for 2D nonlinear triharmonic equations. A popular technique for the biharmonic equation is to split it into two coupled Poisson equations, each of which may be discretised using standard approximations and solved using a Poisson solver. A difficulty with this approach is that the boundary conditions for the new variable Laplacian introduced are not known and need to be approximated at the boundary. Smith [26] and Ehrlich [2, 3] have solved 2D biharmonic equations using coupled second-order accurate finite difference approximations, and Bauer and Riess [1] have used a block iterative method. Kwon *et al.* [7], Stephenson [28], Evans and Mohanty [4], and Mohanty *et al.* [9–12] subsequently developed certain second-order and fourth-order finite difference approximations for biharmonic problems using a 9-point compact cell. The compact cell approach involves discretising the biharmonic equations, using not just the grid values of the unknown solution u but also the values of the derivatives u_{xx}, u_{yy} and u_{zz} at the selected grid points. For 2D and 3D problems, these researchers solved systems of three and four equations to obtain the values of u, u_{xx}, u_{yy} and $u, u_{xx}, u_{yy}, u_{zz}$, respectively. Fourth-order compact finite difference schemes have become quite popular, compared with lower order schemes that require high mesh refinement and hence are less computationally efficient. The higher order accuracy of the fourth-order compact methods, combined with the compactness of the difference stencil, yields highly accurate numerical solutions on relatively coarse grids with greater computational efficiency.

One numerical approach for solving the 2D triharmonic equation (1.1) is to discretise the differential equation on a uniform grid using 49-point approximations with a truncation error of order h^2. This approximation connects central point values, in each case involving 48 neighbouring values of u in a 7×7 grid. The central value of u is connected to grid points three grids away in each direction from the central point, and the difference approximations need to be modified at grid points near the boundaries. However, in the solution of the linear and nonlinear systems obtained through such 49-point discretisation of the 2D triharmonic equation, there are serious computational difficulties that