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An Unconditionally Energy Stable Immersed Boundary Method with Application to Vesicle Dynamics

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Abstract. We develop an unconditionally energy stable immersed boundary method, and apply it to simulate 2D vesicle dynamics. We adopt a semi-implicit boundary forcing approach, where the stretching factor used in the forcing term can be computed from the derived evolutional equation. By using the projection method to solve the fluid equations, the pressure is decoupled and we have a symmetric positive definite system that can be solved efficiently. The method can be shown to be unconditionally stable, in the sense that the total energy is decreasing. A resulting modification benefits from this improved numerical stability, as the time step size can be significantly increased (the severe time step restriction in an explicit boundary forcing scheme is avoided). As an application, we use our scheme to simulate vesicle dynamics in Navier-Stokes flow.

AMS subject classifications: 65M06, 76D07

Key words: Immersed boundary method, unconditionally energy stable, inextensible vesicle, Navier-Stokes flow.

1. Introduction

The immersed boundary (IB) method proposed by Peskin [26] has been successfully applied to many fluid-structure interaction problems — cf. the review [27]. The IB method employs an Eulerian description for the fluid velocity and a Lagrangian description for the configuration of the immersed elastic structure (immersed boundary or interface). The immersed structure exerts some force into the fluid that drives the fluid flow, and at the same time the fluid flow carries the immersed structure to a new configuration. This interaction between the fluid and the immersed structure is linked through a force spreading and velocity interpolating operator, on using a smoothed version of the Dirac delta function [27].

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The IB method is easy to implement and efficient, simply because the immersed structure (no matter how complex) is regarded as a force generator to the fluid, so that the fluid variables can be solved in a fixed Eulerian domain without generating any structure-fitting grid. Many fast efficient fluid solvers can therefore be applied.

Despite substantial success with practical applications using the IB method, it still has some drawbacks from the numerical point of view. Firstly, the method is only first-order accurate, whereas second-order accurate fluid solvers are used. The immersed elastic structure is usually represented one-dimensionally lower than the fluid space so that the exerted force is singular (delta function like), and smoothing the delta function in a regular finite difference scheme causes the method to be first-order accurate only. Although there have been several attempts to improve accuracy, even some including adaptive local mesh refinements near the immersed boundary, formally those methods still remain to be made second-order accurate [6,7,17,22,28].

Another issue is numerical stability. As is well known, the IB method suffers a time step restriction to maintain numerical stability [21,27,29,30]. This restriction becomes more stringent when the elastic force is stiff and the force spreading occurs at the beginning of each time step (an explicit scheme). It is notable that such a time step restriction cannot be alleviated even when the fluid solver is discretised in a semi-implicit manner — i.e. with explicit differencing of the advection term and implicit differencing of the diffusion term. Rather than performing the force spreading at the beginning of the time step, one might consider doing so at an intermediate stage (a semi-implicit scheme) or even at the end of the time step (an implicit scheme). In the past decade, there have been many attempts to reduce the stiffness or to overcome this time step restriction [3, 4, 8, 10, 11, 23, 24]. However, there is always a trade-off between the stability and efficiency of the algorithms involved. In this article, we propose a new semi-implicit scheme that can be solved quite efficiently, where the resultant linear system is symmetric positive definite and the time step size can be significantly increased.

In Section 2, we introduce the formulation for the incompressible Navier-Stokes equations with an immersed elastic interface. We then develop semi-implicit immersed boundary schemes based on the projection method for the fluid solver in Section 3, and show that these developed schemes are unconditionally energy stable. Then we modify these semi-implicit schemes for efficient implementation, with the resultant linear system symmetric positive definite. Numerical results from simulations of vesicle dynamics are given in Section 4, followed by our conclusions and discussion of future work in Section 5.

2. Governing Equations

We begin by stating the mathematical formulation of the Navier-Stokes flow with an immersed boundary (or interface). We consider a moving, immersed, elastic boundary $\Gamma(t)$, which exerts forces into an incompressible fluid in a fixed fluid domain Ω. We assume that the fluids inside and outside of the boundary are the same, so the governing equations