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Pseudo-Tournament Matrices and Their Eigenvalues

Chuanlong Wang[1] and Xuerong Yong[2,*]

[1] *Department of Mathematics, Taiyuan Normal University, Taiyuan, Shanxi, China.*
[2] *Department of Mathematical Sciences, The University of Puerto Rico, Mayaguez, PR 00681, USA.*

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Abstract. A tournament matrix and its corresponding directed graph both arise as a record of the outcomes of a round robin competition. An $n \times n$ complex matrix A is called h-pseudo-tournament if there exists a complex or real nonzero column vector h such that $A + A^* = hh^* - I$. This class of matrices is a generalisation of well-studied tournament-like matrices such as h-hypertournament matrices, generalised tournament matrices, tournament matrices, and elliptic matrices. We discuss the eigen-properties of an h-pseudo-tournament matrix, and obtain new results when the matrix specialises to one of these tournament-like matrices. Further, several results derived in previous articles prove to be corollaries of those reached here.

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1. Introduction

We let X^* and X^t represent the transpose conjugate and the transpose of a vector X, and use the same superscripts $*$ and t to likewise denote the transpose conjugate and transpose of a matrix. An $n \times n$ complex matrix A is called *h-pseudo-tournament* if there is a complex or real nonzero column vector h such that

$$A + A^* = hh^* - I. \tag{1.1}$$

This class of matrices was originally studied by Maybee & Pullman [13], and is a generalisation of the following classes of *tournament-like* matrices satisfying Eq. (1.1) that have received considerable attention in recent decades:

- if A is a real matrix with zero diagonal elements, then A is called an *h-hypertournament* matrix — in this case $h = (h_1, h_2, \cdots, h_n)^t$ where h_j is 1 or -1, $j = 1, \cdots, n$, and their spectral properties were derived [10, 13];

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*Corresponding author. *Email addresses:* `clwang218@126.net` (C. Wang), `xryong@dimacs.rutgers.edu` (X. Yong)

- if A is a nonnegative matrix and $h = \mathit{1}$, where $\mathit{1}$ is the all ones column vector, then A is called a *generalised tournament* matrix [15, 16];
- if A is a zero-one matrix (in this case $h = \mathit{1}$) then A is called a *tournament* matrix — cf. [1, 5, 6, 12, 18] and references therein.

Furthermore, if $-(A+A^*)$ is real then it is an *elliptic* matrix [4, 19], and a real symmetric matrix is elliptic if it has exactly one positive eigenvalue; and $-(A+A^*)$ reduces to a *Householder* matrix if $h^*h = 2$ — cf. Ref. [9]. Incidentally, the techniques we use here are also applicable if the matrix A in Eq. (1.1) satisfies $A + A^* = -hh^* - I$. Without loss of generality, we assume throughout our discussion that h has no zero element.

A tournament matrix and its corresponding directed graph both arise as a record of the outcomes of a round robin competition. The need and desire to come up with player ranking schemes has motivated an extensive study of the combinatorial and spectral properties of tournament matrices and their generalisations. Hypertournament matrices, the generalised tournament matrices, and pseudo-tournament matrices can be understood as weighted tournaments. They not only provide a means for inquiring into the properties of more general tournaments but also are the source of matrix analytic challenges of independent interest, which interplay between matrix/graph theoretic and spectral properties. There is a wealth of literature that focuses on deriving algebraic or combinatorial attributes of these matrices [1, 2, 5, 10, 15, 18]. In particular, Brauer & Gentry [1, 2] showed that $-1/2 \leq \operatorname{Re}\lambda \leq (n-1)/2$ and $|\operatorname{Im}\lambda| \leq \sqrt{n(n-1)/6}$ if λ is an eigenvalue of a tournament matrix A of order n. Moon & Pullman [16] then proved that similar results also hold for the generalised tournament matrices. Subsequently, Maybee & Pullman [13] considered the more general pseudo-tournament and h-hypertournament matrices, and proved the inequality $-1/2 \leq \operatorname{Re}\lambda \leq (n-1)/2$ for the h-hypertournament matrices. It is notable that any h-hypertournament matrix A is diagonally and orthogonally similar to a $\mathit{1}$-hypertournament matrix, because we then have $Dh = \mathit{1}$ where $h = (h_1, h_2, \cdots, h_n)^t$ and $D = diag(h_1, h_2, \cdots, h_n)$ with $h_i = 1$ or -1 $\forall\, i$, such that $D^*(A+A^*)D = \mathit{1}\mathit{1}^t - I$). Accordingly, any investigation of the eigen-properties of an h-hypertournament matrix is equivalent to working on the eigen-properties of a $\mathit{1}$-hypertournament matrix.

If A is an $n \times n$ $\mathit{1}$-hypertournament matrix then $s = A\mathit{1}$ is called the *score vector* of A, and if $s = ((n-1)/2)\mathit{1}$ then A is said to be *regular*. The score vector s plays an important role for the eigenvalues of these matrices [10, 13]. Any $\mathit{1}$-hypertournament matrix satisfies $s^t\mathit{1} = n(n-1)/2$ and $s^ts \geq n(n-1)^2/4$, with equality if and only if it is regular. Here we introduce similar definitions: for an $n \times n$ h-pseudo-tournament matrix A, we call $s = Ah$ the *pseudo-score vector* of A, and say A is *pseudo-regular* if $Ah = (h^*h - 1/2)h$. We note that a regular $\mathit{1}$-hypertournament matrix is a $\mathit{1}$-pseudo-regular tournament matrix; and also say that a $2n \times 2n$ $\mathit{1}$-hypertournament matrix T is *almost regular* if it has n row sums equal to $n-1$ and n row sums equal to n. These definitions will be used in our discussion on localising the eigenvalues of an h-pseudo-tournament matrix. We also use the following notation:

$\mathbf{C}^n(\mathbf{R}^n)$:	the n-dimensional complex (real) Euclidean vector space
$\lambda_i(A)$:	the ith eigenvalue of matrix A; sometimes, write $\lambda_i(A)$ simply as λ_i

$\mathrm{Re}\,\lambda_i(A)$:	the real part of $\lambda_i(A)$
$\mathrm{Im}\,\lambda_i(A)$:	the imaginary part of $\lambda_i(A)$
$J = 11^t$:	the all one's matrix of an appropriate size
$\rho(A)$:	the spectral radius of matrix A (the Perron value, if A is nonnegative)
$\|\cdot\|_2$:	the Euclidean norm

In addition, for an $n \times n$ matrix $A \in \mathbf{C}^n$ we assume that

$$\mathrm{Re}\,\lambda_1(A) \geq \mathrm{Re}\,\lambda_2(A) \geq \cdots \geq \mathrm{Re}\,\lambda_n(A)\,.$$

In brief, the aim of this article is to derive more general and comprehensive properties of h-pseudo-tournament matrices. We describe some preliminaries and fundamentals in Section 2, and in Section 3 derive the new properties. In particular, when the h-pseudo-tournament matrix reduces to one of the above-mentioned tournament-like matrices, we obtain some new results. Further, results previously obtained for tournament-like matrices appear valid for h-pseudo-tournament matrices. We also generalise some known results, and determine new algebraic properties for almost regular tournament matrices.

2. Preliminaries and Lemmas

Lemma 2.1. *Let $x = (x_1, x_2, \cdots, x_n)^t$ and $y = (y_1, y_2, \cdots, y_n)^t$ be two vectors in $\mathbf{R}^n$ such that*

$$x_1 \geq x_2 \geq \cdots \geq x_n\,, \qquad y_1 \geq y_2 \geq \cdots \geq y_n\,.$$

Then the following four statements are equivalent:

(a) $y = Sx$ *for a doubly stochastic matrix* S*;*

(b) $\sum_{i=1}^k x_i \geq \sum_{i=1}^k y_i$, $k = 1, 2, \cdots, n-1$, *and* $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$*;*

(c) $\sum_{i=1}^n \phi(x_i) \geq \sum_{i=1}^n \phi(y_i)$, *for any continuous convex function* ϕ*; and*

(d) there exists an $n \times n$ *Hermitian matrix with eigenvalues* $x_1, x_2, \cdots, x_n$ *and diagonal elements* $y_1, y_2, \cdots, y_n$.

Proof. A proof of the equivalence of statements (a), (b), and (c) is given in Ref. [7]; and the equivalence of statements (a) and (d) is proven in Refs. [8,14]. □

Lemma 2.2. *Let A be an $n \times n$ irreducible zero-one symmetric matrix with zero diagonal. Then $\lambda_n(A) \leq -1$ and the following three statements are equivalent:*

(a) $\lambda_n(A) = -1$*;*

(b) $A = J - I$*; and*

(c) $\lambda_1(A) = n - 1$, *which implies* $\lambda_j(A) = -1$, $\forall j > 1$.

Proof. The lemma follows directly from the well-known Cauchy Interlacing Theorem [9], and a proof is given in Ref. [20]. □

Lemma 2.3. *Let T be an $2n \times 2n$ almost regular 1-hypertournament matrix. Then*

$$\rho(T) \geq \frac{n-1}{2} + \frac{n}{2}\sqrt{1 - \frac{1}{n^2}} \, .$$

Proof. Since A is almost regular, its score vector s has n entries equal to n and n entries equal to $n-1$, so $s^t s = n(2n^2 - 2n + 1)$ and the result follows directly from Theorem 1 of Ref. [10]. □

Any h-hypertournament matrix A, $\lambda_1(A)$ is real positive and satisfies $\rho(A) = \lambda_1(A)$ — cf. [10]. However, this property is not always valid for h-pseudo-tournament matrices. For example, if

$$A = \begin{pmatrix} 2i & -2 \\ 0 & \frac{3}{2} + i \end{pmatrix}$$

then

$$A + A^* = \begin{pmatrix} 0 & -2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} - I = hh^* - I \quad \text{where } h = \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

so A is an h-pseudo-tournament matrix — but $\lambda_1(A) = (3/2) + i$ and $|\lambda_1(A)| < \rho(A) = |2i| = 2$. In the next section, we establish a theorem where $\rho(A) = \lambda_1(A)$ for an h-pseudo-tournament matrix A.

The following lemma bounds the partial sums of the real parts of the eigenvalues of an h-pseudo-tournament matrix.

Lemma 2.4. *Let A be an $n \times n$ h-pseudo-tournament matrix. Then there exists a collection of numbers $q_1, q_2, \cdots, q_n$ satisfying $1 \geq q_1 \geq q_2 \geq \cdots \geq q_n \geq 0$ and $\sum_j q_j = 1$ such that*

$$2\,\mathrm{Re}\,\lambda_1(A) = h^*hq_1 - 1\,, \quad 2\,\mathrm{Re}\,\lambda_2(A) = h^*hq_2 - 1\,, \quad \cdots, \quad 2\,\mathrm{Re}\,\lambda_n(A) = h^*hq_n - 1\,,$$

implying

$$\sum_{i=1}^{k} \mathrm{Re}\,\lambda_i(A) \leq \frac{h^*h - k}{2}, \quad k = 1, 2, \cdots, n-1\,,$$
$$\sum_{i=1}^{n} \mathrm{Re}\,\lambda_i(A) = \frac{h^*h - n}{2}\,,$$

and in particular $-1/2 \leq \mathrm{Re}\,\lambda_n(A)$, $\mathrm{Re}\,\lambda_1(A) \leq h^*h - 1/2$.

Proof. There exists a unitary matrix Q such that (cf. Schur Theorem [9])

$$Q^*AQ = \begin{pmatrix} \lambda_1(A) & & a_{ij} \\ & \ddots & \\ 0 & & \lambda_n(A) \end{pmatrix}, \quad \text{so} \quad Q^*(A + A^*)Q = \begin{pmatrix} 2\,\mathrm{Re}\,\lambda_1(A) & & a_{ij} \\ & \ddots & \\ a_{ij}^* & & 2\,\mathrm{Re}\,\lambda_n(A) \end{pmatrix}.$$

Since $\lambda(A+A^*) = \lambda(Q^*(A+A^*)Q)$, from Lemma 2.1(b) we have

$$\sum_{i=1}^{k} 2\,\mathrm{Re}\,\lambda_i(A) \le \sum_{i=1}^{k} \lambda_i(A+A^*)\,, \quad k = 1, \cdots, n-1\,,$$
$$\sum_{i=1}^{n} 2\,\mathrm{Re}\,\lambda_i(A) = \sum_{i=1}^{n} \lambda_i(A+A^*).$$

The eigenvalues of $A+A^* = hh^* - I$ are

$$\lambda_1(A+A^*) = h^*h - 1\,,$$
$$\lambda_j(A+A^*) = -1, \quad j = 2, \cdots, n\,,$$

so from Lemma 2.1(a) there is a doubly stochastic matrix S such that

$$\begin{pmatrix} 2\,\mathrm{Re}\,\lambda_1(A) \\ \vdots \\ 2\,\mathrm{Re}\,\lambda_n(A) \end{pmatrix} = S \begin{pmatrix} h^*h-1 \\ \vdots \\ -1 \end{pmatrix} = h^*h \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

where $(q_1, q_2, \cdots, q_n)^t$ is the first column of S, hence

$$2\,\mathrm{Re}\,\lambda_1(A) = h^*hq_1 - 1\,,$$
$$2\,\mathrm{Re}\,\lambda_2(A) = h^*hq_2 - 1\,,$$
$$\cdots,$$
$$2\,\mathrm{Re}\,\lambda_n(A) = h^*hq_n - 1\,,$$

where $\sum_{i=1}^{n} q_i = 1$ and $1 \ge q_1 > q_2 \ge \cdots \ge q_n \ge 0$. From these relations, it follows that $\mathrm{Re}\,\lambda_j(A) \ge -1/2$, and hence

$$\sum_{i=1}^{k} \mathrm{Re}\,\lambda_i(A) \le \frac{h^*h-k}{2}, \quad k = 1, 2, \cdots, n-1\,,$$
$$\sum_{i=1}^{n} \mathrm{Re}\,\lambda_i(A) = \frac{h^*h-n}{2}\,.$$

In particular, the cases $k = 1$ and n generate

$$\mathrm{Re}\,\lambda_1(A) \le \frac{h^*h-1}{2}\,, \qquad \mathrm{Re}\,\lambda_n(A) \ge -\frac{1}{2}\,. \qquad \square$$

From Lemma 2.4, for an $n \times n$ h-pseudo-tournament matrix A we have the equality

$$\mathrm{Re}\,\lambda_k(A) + \mathrm{Re}\,\lambda_{k-1}(A) + \cdots + \mathrm{Re}\,\lambda_1(A) = \frac{h^*h-k}{2}, \qquad \forall\ k \ge 1$$

if and only if A has at least $n-k$ eigenvalues with real parts equal to $-1/2$ — $\mathrm{Re}\,\lambda_j(A) = -1/2$, $j = k+1, \cdots, n$. For an *1*-hypertournament matrix A, Kirkland [10] derives an *if and only if* property for which $\mathrm{Re}\,\lambda_2(A) + \rho(A) = (n-2)/2$ in (Theorem 2), and subsequently Kirkland & Shader [11] derive equivalent properties for a tournament matrix that satisfies $\rho(A) = (n-k)/2$ and $\lambda_j(A) = 0$, $j = 2, 3, \cdots, k$ (Theorem 3). By modifying their ideas, similar results can be reached for h-pseudo-tournament matrices.

Corollary 2.1. *Let A be an $n \times n$ h-hypertournament matrix. Then*

$$\sum_{i=1}^{k} \mathrm{Re}\,\lambda_i(A) \le \frac{n-k}{2}, \quad k = 1, 2, \cdots, n-1\,, \quad \sum_{i=1}^{n} \mathrm{Re}\,\lambda_i(A) = 0\,.$$

In particular, these inequalities imply [13]

$$-1/2 \le \mathrm{Re}\,\lambda_n(A)\,, \quad \rho(A) \le (n-1)/2\,.$$

Proof. From the definition of the h-hypertournament matrices we have $h^*h = n$. On the other hand, it is known that $\rho(A) = \lambda_1(A)$ for an h-hypertournament matrix A [10]. The rest of the proof follows trivially from Lemma 2.4. □

From Lemma 2.4, if A is an $n \times n$ h-pseudo-tournament matrix then

$$\mathrm{Re}\,\lambda_1(A) \ge \frac{h^*h-2}{4} + p \ \text{ implies } \sum_{i=2}^{j} \mathrm{Re}\,\lambda_i(A) \le \frac{h^*h-2j+2}{4} - p\,, \ 2 \le j \le n \tag{2.1}$$

where p is any real number. We use this property in bounding the partial sums of the real parts of the eigenvalues of A later. It is of interest to consider h-pseudo-tournament matrices such that $\mathrm{Re}\,\lambda_1(A) > (h^*h-2)/2$. Corollary 1.2 of Ref. [10] states that if A is an $n \times n$ *1*-hypertournament matrix satisfying $s^t s \le n^2(n-1)/4$ (which implies $\rho(A) > (n-2)/2$) then A has exactly one real positive eigenvalue, and all other eigenvalues have negative real parts — its determinant therefore has sign $(-1)^{n-1}$. Furthermore, if A is a generalised tournament matrix then it is also primitive [10]. The following corollary implies that the same statements are also true for an h-pseudo-tournament matrix A, if $\lambda_1(A)$ is real and greater than $(h^*h-2)/2$.

Corollary 2.2. *Assume A is a real $n \times n$ h-pseudo-tournament matrix, $\lambda_1(A)$ is real and greater than $(h^*h-2)/2$, and $h^*h \ge 2$. Then*

(a) A has exactly one real positive eigenvalue and all other eigenvalues have negative real parts, so its determinant has sign $(-1)^{n-1}$; and

(b) if $n > 3$ and A is nonnegative, then A is primitive.

Proof. From Lemma 2.4, we have that

$$\mathrm{Re}\,\lambda_2(A) + \lambda_1(A) \le \frac{h^*h-2}{2}\,,$$

so $\lambda_1(A) > (h^*h-2)/2$ implies $\operatorname{Re}\lambda_2(A) < 0$. Since the number of complex eigenvalues with nonzero imaginary part is even, A has exactly one real positive eigenvalue and all of its other eigenvalues have negative real parts — this proves part (a).

We now apply the Perron-Frobenius theorem in nonnegative matrix theory to show (b). Since A is nonnegative, $h = (h_1, h_2, \cdots, h_n)^t$ is positive and $h_i \geq 1$, $\forall\, i$. Let A_i be any $(n-1)\times(n-1)$ principal sub-matrix of A. Then $\rho(A_i) \leq \rho((A_i + A_i^t)/2) \leq (h^t h - 2)/2$ (since $h_i^2 \geq 1$, $\forall i$), so $\rho(A) > (h^t h-2)/2$ implies that A must be irreducible. Now, if A were non-primitive, for some permutation matrix P we would have

$$P^tAP = \begin{pmatrix} 0 & B & 0 \\ 0 & 0 & C \\ D & 0 & 0 \end{pmatrix} \quad \text{or} \quad P^tAP = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

since $A + A^t = hh^t - I$ has no zero off-diagonal elements, implying that B, C, D must be positive numbers and $n \leq 3$. This contradicts the assumption that $n > 3$, and the proof of part (b) follows. ☐

Kirkland [10] proved that if A is an $2n\times 2n$ almost regular *1*-hypertournament matrix then $\operatorname{Re}\lambda_2(A) \leq -\frac{1}{2} + \frac{1}{2n}\,\frac{1}{1+\sqrt{1-1/n^2}}$ (a rewritten form), so A has exactly one real positive eigenvalue and an odd number of real negative eigenvalues, and its all other eigenvalues have negative real parts (his Corollary 1.4) . The following corollary extending this inequality is subsequently applied in establishing our theorems.

Corollary 2.3. *Let A be an* $2n\times 2n$ *almost regular 1-hypertournament matrix. Then the real parts of the eigenvalues* $\operatorname{Re}\lambda_j(A), j \geq 2$ *satisfy*

$$-\frac{j-1}{2} \leq \sum_{i=2}^{j} \operatorname{Re}\lambda_i(A) \leq -\frac{j-1}{2} + \frac{1}{2n}\,\frac{1}{1+\sqrt{1-1/n^2}}\,, \qquad 2 \leq j \leq 2n\,.$$

(*In particular, when* $j=2$ *we have* $\operatorname{Re}\lambda_2(A) \leq -\frac{1}{2} + \frac{1}{2n}\,\frac{1}{1+\sqrt{1-1/n^2}}$.)

Proof. The first part of the inequality is trivial because $\operatorname{Re}\lambda_i(A) \geq -1/2$, so it remains to prove the second. From Corollary 2.1, $\forall j \geq 2$ we have

$$\sum_{i=2}^{j} \operatorname{Re}\lambda_i(A) + \lambda_1(A) \leq \frac{2n-j}{2}\,.$$

From Lemma 2.3, we see that the number p in (2.1) can be taken as $(n/2)\sqrt{1-1/n^2}$, implying

$$\begin{aligned}\sum_{i=2}^{j} \operatorname{Re}\lambda_i(A) &\leq -\frac{2j-2}{4} + \frac{n}{2} - \frac{n}{2}\left(\sqrt{1-1/n^2}\right) \\ &= -\frac{j-1}{2} + \frac{1}{2n}\,\frac{1}{1+\sqrt{1-1/n^2}}\,. \quad \square\end{aligned}$$

Maybee & Pullman [13] (Theorem 3.1) showed that if $\lambda(A)$ is an eigenvalue of an h-pseudo-tournament matrix A then $\operatorname{Re}\lambda = -1/2$ or $rank(A-\lambda I) = n-1$. In the following lemma, we reduce an h-pseudo-tournament matrix by the Schur decomposition theorem to get an upper triangular matrix, from which it similarly follows that the geometric multiplicity of any of its eigenvalues with real part unequal to $-1/2$ is 1.

Lemma 2.5. *Let A be an $n \times n$ h-pseudo-tournament matrix having $n-n_0$ eigenvalues with real parts equal to $-1/2$, $1 \le n_0 \le n$. Then A is similar to the upper triangular matrix $U \oplus V$, where*

$$U = \begin{pmatrix} \lambda_1(A) & 2\operatorname{Re}\lambda_1(A)+1 & 2\operatorname{Re}\lambda_1(A)+1 & \cdots & 2\operatorname{Re}\lambda_1(A)+1 \\ 0 & \lambda_2(A) & 2\operatorname{Re}\lambda_2(A)+1 & \cdots & 2\operatorname{Re}\lambda_2(A)+1 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n_0}(A) \end{pmatrix}$$

and $V = diag(\lambda_{n_0+1}(A), \lambda_{n_0+2}(A), \cdots, \lambda_n(A))$ is a diagonal matrix with $\operatorname{Re}\lambda_{n_0+i}(A) = -1/2$, $1 \le i \le n-n_0$. In particular, if A has no eigenvalue with real part equal to $-1/2$, then A is non-derogatory (i.e. each eigenvalue has geometric multiplicity equal to 1).

Proof. There exists a unitary matrix Q, such that Q^*AQ is given by the following upper triangular structure:

$$Q^*AQ = \begin{pmatrix} \lambda_1(A) & & a_{ij} \\ & \ddots & \\ 0 & & \lambda_n(A) \end{pmatrix}$$

such that

$$Q^*(A+A^*)Q + I = \begin{pmatrix} 2\operatorname{Re}\lambda_1(A)+1 & & a_{ij} \\ & \ddots & \\ a^*_{ij} & & 2\operatorname{Re}\lambda_n(A)+1 \end{pmatrix},$$

where a^*_{ij} are the conjugates of a_{ij} $\forall\, i,j,\ i \neq j$. Since $Q^*(A+A^*)Q + I = Q^*hh^*Q$, the rank of Q is 1 — and hence for all i,j it follows that

$$a_{ij} = \sigma^*_i \sigma_j \sqrt{(2\operatorname{Re}\lambda_i(A)+1)(2\operatorname{Re}\lambda_j(A)+1)}\,,$$

where σ_i are the complex numbers of modulus 1. Now defining $D = diag(\sigma_1, \cdots, \sigma_n)$, we have $(QD)^*AQD = R$ where

$$R = \begin{pmatrix} \lambda_1(A) & \sqrt{(2\operatorname{Re}\lambda_1(A)+1)(2\operatorname{Re}\lambda_2(A)+1)} & \dots & \sqrt{(2\operatorname{Re}\lambda_1(A)+1)(2\operatorname{Re}\lambda_n(A)+1)} \\ 0 & \lambda_2(A) & \dots & \sqrt{(2\operatorname{Re}\lambda_2(A)+1)(2\operatorname{Re}\lambda_n(A)+1)} \\ \vdots & \ddots & \ddots & \\ 0 & 0 & \dots & \lambda_n(A) \end{pmatrix}. \tag{2.2}$$

If A has $n-n_0$ eigenvalues with real part equal to $-1/2$, then $2\operatorname{Re}\lambda_i(A)+1 > 0$ $\forall\, i \le n_0$, and $2\operatorname{Re}\lambda_i(A)+1 = 0$ for $n_0 < i \le n$. On defining

$$D_1^{-1} = diag(\sqrt{2\operatorname{Re}\lambda_1(A)+1}\,, \quad \cdots, \quad \sqrt{2\operatorname{Re}\lambda_{n_0}(A)+1},\ 1, \cdots, 1)\,,$$

we have $D_1^{-1}D^*Q^*AQDD_1 = (QDD_1)^{-1}A(QDD_1) = U \oplus V$, where

$$U = \begin{pmatrix} \lambda_1(A) & 2\operatorname{Re}\lambda_1(A)+1 & 2\operatorname{Re}\lambda_1(A)+1 & \cdots & 2\operatorname{Re}\lambda_1(A)+1 \\ 0 & \lambda_2(A) & 2\operatorname{Re}\lambda_2(A)+1 & \cdots & 2\operatorname{Re}\lambda_2(A)+1 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n_0}(A) \end{pmatrix}$$

and $V = diag(\lambda_{n_0+1}(A), \lambda_{n_0+2}(A), \cdots, \lambda_n(A))$, with $\operatorname{Re}\lambda_{n_0+i}(A) = -1/2$, $1 \le i \le n - n_0$. As $1 \le i \le n_0$ for all i, $2\operatorname{Re}\lambda_i(A)+1 > 0$ and the rank of $U - \lambda_i(A)I$ is $n_0 - 1$. The dimension of the eigenspace of U corresponding to each $\lambda_i(A)$ is therefore 1, so U is non-derogatory. □

Observing the matrix U in Lemma 2.5 we see that the rank of an h-pseudo-tournament matrix A is at least $n-1$. Moreover, if A has an eigenvalue with real part equal to $1/2$, then the algebraic and geometric multiplicities of this eigenvalue are the same. Gaen *et al.* proved that, if an *1*-hypertournament matrix A has an eigenvalue with real part equal to $-1/2$, the algebraic and geometric multiplicities of this eigenvalue are the same — cf. Ref. [3]. From Lemma 2.5, for h-pseudo-tournament matrices, one can establish results similar to those given in Lemma 1 of Ref. [11] in almost the same way.

Maybee & Pullman [13] also showed that, if A is an h-pseudo-tournament matrix with pure imaginary diagonal elements (in this case $h^*h = n$), then $\rho(A) = \lambda_1(A) = (n-1)/2$ if and only if A is regular; and when this so, $\operatorname{Re}\lambda_j(A) = -1/2\ \forall\, j > 1$. The following corollary provides analogous statements about an h-pseudo-tournament matrix A with the property $\operatorname{Re}\lambda_1(A) = (h^*h-1)/2$.

Corollary 2.4. *For an $n \times n$ h-pseudo-tournament matrix A, the the following statements are equivalent:*

(a) $\operatorname{Re}\lambda_1(A) = (h^*h-1)/2$;

(b) $\operatorname{Re}\lambda_2(A) = -1/2$;

(c) A is normal; and

(d) $Ah = ((h^*h-1)/2 + bi)h,\ i = \sqrt{-1}$.

Proof. The proof simply involves combining Lemma 2.4 with (2.2) derived in the proof of Lemma 2.5. □

In this corollary, if A is a real matrix then h can be a real or a pure imaginary vector, so from part (d) we have $\lambda_1(A) = (h^*h-1)/2$. If A is an $n \times n$ h-pseudo-tournament matrix and $h^*\mathit{1}$ is real, we may define the Householder matrix H (a unitary Hermitian matrix) as [9]

$$H = I - 2\omega\omega^* \quad \text{where} \quad \omega = \frac{h - \sqrt{h^*h/n}\,\mathit{1}}{\|h - \sqrt{h^*h/n}\,\mathit{1}\|_2}$$

such that $Hh = \sqrt{h^*h/n}\,1$, where we set $\omega = 0$ if $h = \sqrt{h^*h/n}\,1$. Then

$$Hhh^*H^* = Hh(Hh)^* = \frac{h^*h}{n}11^t \,,$$

so that

$$HAH^* + HA^*H^* = H(A + A^*)H^* = \frac{h^*h}{n}J - I \,. \tag{2.3}$$

Consequently, if A is an $n \times n$ h-pseudo-tournament matrix and h^*1 is real then A is orthogonally similar to HAH^*, which is a $\sqrt{h^*h/n}\ \ 1$-pseudo-tournament matrix. Further, if $h^*h = n$ then HAH^* is a 1-pseudo-tournament matrix.

Lemma 2.6. *Suppose A is a 1-pseudo-tournament matrix of order n, and that* $\mathrm{Re}\,\lambda_{k+1}(A) = -1/2$, $k \geq 1$ *(for smallest k if there exist more than one such k). The rank of* $(1, A1, \cdots, A^{k-1}1)$ *is then k.*

Proof. The proof is quite similar to that of Theorem 1 in Ref. [11]. □

3. The Perron Vector and Eigenvalues of an h-Pseudo-Tournament Matrix

In this section, we call the eigenvector r corresponding to $\lambda_1(A)$ the Perron vector of an h-pseudo-tournament matrix A, and for simplicity abbreviate $\lambda_j(A)$ as λ_j. The formula (2.3) will be applied to discuss the eigen-properties of matrix A.

Theorem 3.1. *Let A be a 1-pseudo-tournament matrix of order n and its eigenvalues satisfy* $\mathrm{Re}\,\lambda_{k+1} = -1/2$, $k \geq 1$ *(for smallest k if there exist more than one such k). Then its Perron vector is*

$$\begin{aligned} r = (\lambda_2\lambda_3\cdots\lambda_k)1 - &\left(\sum_{2\leq j_1<\cdots<j_{k-2}\leq k} \lambda_{j_1}\lambda_{j_2}\cdots\lambda j_{k-2}\right) A1 \\ &+ \left(\sum_{2\leq j_1<\cdots<j_{k-3}\leq k} \lambda_{j_1}\lambda_{j_2}\cdots\lambda j_{k-3}\right) A^2 1 - \cdots + (-1)^{k-1}A^{k-1}1 \,. \end{aligned}$$

Conversely, if for some α_j the Perron vector is

$$r = \alpha_1 1 + \alpha_2 A1 + \cdots + \alpha_2 A^{k-1}1$$

and the rank of $(1, A1, \cdots, A^{k-1}1)$ *is k, then* $\mathrm{Re}\,\lambda_{k+1} = -1/2$.

Proof. Let $\mu_j = \sqrt{2\,\mathrm{Re}\,\lambda_j + 1}$, $j = 1, 2, \cdots, n$. Assume that $Q_1 = (q_1, q_2, \cdots, q_n)$ ($Q_1 = QD$) is the unitary matrix such that $Q_1^* A Q_1 = R$, where R is the triangular matrix (2.2) obtained in the proof of Lemma 2.5. Since R is the direct sum of an upper triangular matrix R_k and a diagonal matrix V, and since $\mathrm{Re}\,\lambda_{k+1} = \cdots = \mathrm{Re}\,\lambda_n = -1/2$, we have

$$\begin{aligned}
A+A^* =&(q_1,q_2,\cdots,q_n)(R+R^*)(q_1,q_2,\cdots,q_n)^* \\
=&2(\mathrm{Re}\,\lambda_1)q_1q_1^* + 2(\mathrm{Re}\,\lambda_2)q_2q_2^* + \cdots + 2(\mathrm{Re}\,\lambda_k)q_kq_k^* - (q_{k+1}q_{k+1}^* + \cdots + q_nq_n^*) \\
&+ \sum_{1\le i<j\le k} \mu_i\mu_j(q_iq_j^* + q_jq_i^*) \\
=&(2\,\mathrm{Re}\,\lambda_1+1)q_1q_1^* + (2\,\mathrm{Re}\,\lambda_2+1)q_2q_2^* + \cdots + (2\,\mathrm{Re}\,\lambda_k+1)q_kq_k^* \\
&- (q_1q_1^* + \cdots + q_nq_n^*) + \sum_{1\le i<j\le k} \mu_i\mu_j(q_iq_j^* + q_jq_i^*) \\
=&J - I \,,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\mathit{1}\mathit{1}^t =&\mu_1^2q_1q_1^* + \mu_2^2q_2q_2^* + \cdots + \mu_k^2q_kq_k^* + \sum_{1\le i<j\le k} \mu_i\mu_j(q_iq_j^* + q_jq_i^*) \\
=&(\mu_1q_1 + \mu_2q_2 + \cdots + \mu_kq_k)(\mu_1q_1 + \mu_2q_2 + \cdots + \mu_kq_k)^*.
\end{aligned}$$

By appropriately selecting the unitary vectors q_js we therefore have

$$\mathit{1} = \mu_1q_1 + \mu_2q_2 + \cdots + \mu_kq_k\,.$$

Now for $j = 1,2,\cdots,k-1$, applying the relation $(A^jq_1,A^jq_2,\cdots,A^jq_n) = (q_1,q_2,\cdots,q_n)R^j$,

$$\begin{aligned}
A^j\mathit{1} =&\mu_1A^jq_1 + \mu_2A^jq_2 + \cdots + \mu_kA^jq_k \\
=&(A^jq_1,A^jq_2,\cdots,A^jq_k)\begin{pmatrix}\mu_1\\ \mu_2\\ \vdots\\ \mu_k\end{pmatrix} \\
=&(q_1,q_2,\cdots,q_k)R_k^j v\,,
\end{aligned}$$

where $v = (\mu_1,\mu_2,\cdots,\mu_k)^t$ and

$$R_k = \begin{pmatrix}\lambda_1 & \mu_1\mu_2 & \dots & \mu_1\mu_k\\ 0 & \lambda_2 & \dots & \mu_2\mu_k\\ \vdots & \ddots & \ddots & \\ 0 & 0 & \dots & \lambda_k\end{pmatrix}$$

is the $k\times k$ leading principal sub-matrix of R. Consequently,

$$(\mathit{1},A\mathit{1},A^2\mathit{1},\cdots,A^{k-1}\mathit{1}) = (q_1,q_2,\cdots,q_k)(v,R_kv,R_k^2v,\cdots,R_k^{k-1}v)\,, \tag{3.1}$$

From Lemma 2.6, the rank of $(\mathit{1},A\mathit{1},A^2\mathit{1},\cdots,A^{k-1}\mathit{1})$ is k, so the $k\times k$ matrix

$$(v,R_kv,R_k^2v,\cdots,R_k^{k-1}v)$$

is nonsingular. We now determine the first column of the inverse of $(v, R_k v, R_k^2 v, \cdots, R_k^{k-1} v)$, in order to express q_1 in terms of the vectors 1, $A1$, $A^2 1$, $\cdots$, $A^{k-1}1$. To this end, let

$$R_k = \begin{pmatrix} \lambda_1 & \alpha \\ 0 & R_{k-1} \end{pmatrix}, \qquad \alpha = (\mu_1\mu_2, \ \cdots, \ \mu_1\mu_k) .$$

The eigenvalues of R_{k-1} are $\lambda_2, \lambda_3, \cdots, \lambda_k$, and for $j = 1, 2, \cdots, k-1$ we have

$$R_k^j = \begin{pmatrix} \lambda_1^j & * \\ 0 & R_{k-1}^j \end{pmatrix} .$$

Then letting $u = \left(p_k(0), p_k'(0), \cdots, \frac{p_k^{(k-1)}(0)}{(k-1)!} \right)^t$, where

$$p_k(\lambda) = (\lambda_2 - \lambda)(\lambda_3 - \lambda)\cdots(\lambda_k - \lambda)$$

is the characteristic polynomial of R_{k-1}, on using the Hamiltonian theorem ($p_k(R_{k-1}) = 0$) we obtain

$$(v, R_k v, R_k^2 v, \cdots, R_k^{k-1} v)\, u = \begin{pmatrix} c \\ 0 \end{pmatrix} .$$

Since $\mu_j > 0$, $c \neq 0$ and u/c is the first column of $(v, R_k v, R_k^2 v, \cdots, R_k^{k-1} v)^{-1}$, from (3.1) we have

$$q_1 = \frac{1}{c} \left(p_k(0) 1 + p_k'(0) A1 + \cdots + \frac{p_k^{(k-1)}(0)}{(k-1)!} A^{k-1} 1 \right) .$$

Finally, applying the relations

$$\frac{p_k^{(j)}(0)}{j!} = (-1)^j \sum_{2 \le j_1 < \cdots < j_{k-j-1} \le k} \lambda_{j_1} \lambda_{j_2} \cdots \lambda j_{k-j-1}$$

and removing the constant c in q_1 yields the formula for the Perron vector r in the theorem.

Conversely, if for some α_j we have

$$r = \alpha_1 1 + \alpha_2 A1 + \cdots + \alpha_2 A^{k-1} 1$$

and the rank of $(1, A1, \cdots, A^{k-1}1)$ is k, then from the above discussion and $\operatorname{Re}\lambda_{k+1} \ge -1/2$ it is easy to prove by contradiction that $\operatorname{Re}\lambda_{k+1} = -1/2$. □

Thus if $\operatorname{Re}\lambda_2 = -1/2$ then $r = 1$ and hence A is a regular 1-pseudo-tournament matrix, which coincides with Lemma 2.5.

Corollary 3.1. *Let A is a 1-pseudo-tournament matrix of order n. Then* $\operatorname{Re}\lambda_1 + \operatorname{Re}\lambda_2 = (n-2)/2$ *if and only if its Perron vector r can be taken as*

$$r = \lambda_2 1 - A1 .$$

Thus if A is a tournament matrix, λ_1 and λ_2 must real numbers; and because $r = \lambda_2 \mathit{1} - A\mathit{1}$, from the relation $Ar = \lambda_1 r$ it can be checked easily that

$$\lambda_1\lambda_2 \mathit{1} - (\lambda_1 + \lambda_2)A\mathit{1} + A^2\mathit{1} = 0 ,$$

which implies that $\lambda_1 = (n-2)/4 + \sqrt{b}$ and $\lambda_2 = (n-2)/4 - \sqrt{b}$ where $b \geq 0$ is an integer.

Kirkland & Shader [11] derived the following Corollary 3.2 for a tournament matrix A with zero eigenvalues.

Corollary 3.2. *Let A be a 1-pseudo-tournament matrix of order n and $\lambda_j = 0$ for $j = 2, 3, \cdots, k$. Then $\mathrm{Re}\,\lambda_1 = (n-k)/2$ if and only if its Perron vector r can be taken as $r = A^{k-1}\mathit{1}$.*

Proof. It is a direct corollary to Theorem 3.1. □

Remark 3.1. For an h-pseudo-tournament matrix A of order n, if $\mathrm{Re}\,\lambda_{k+1} = -1/2$ (where $k \geq 1$ is the smallest index if there are more than one such k), then its Perron vector can be taken as Hr where H is the Householder matrix given in (2.3). The converse also holds.

The following theorem provides a lower-bound λ_1 and upper-bound partial sums of $\mathrm{Re}\,\lambda_j$ for $j \geq 2$.

Theorem 3.2. *Let A be an $n \times n$ h-pseudo-tournament matrix whose pseudo-score vector $s = Ah = (s_1, s_2, \cdots, s_n)^t$ satisfies*

$$\sum_{i<j} |s_i\bar{h_j} - \bar{s_j}h_i|^2 < \left(\frac{h^*h}{2} - \left(\frac{h^*h}{2}\right)^2 + \mathrm{Re}\,h^*s\right)^2 \tag{3.2}$$

where $h = (h_1, \cdots, h_n)^t \neq 0$. Then A has exactly one real positive eigenvalue λ_1 such that

$$\lambda_1 \geq \frac{-\left(\frac{h^*h}{2}\right)^2 + \mathrm{Re}\,h^*s + \sqrt{\left(\frac{h^*h}{2} - \left(\frac{h^*h}{2}\right)^2 + \mathrm{Re}\,h^*s\right)^2 - \sum_{i<j}|s_i\bar{h_j} - \bar{s_j}h_i|^2}}{h^*h} ,$$

while for $\forall j \geq 2$ the real parts of its other eigenvalues satisfy

$$\sum_{i=2}^{j} \mathrm{Re}\,\lambda_i \leq \frac{3h^*h - 2j}{4} - \frac{\mathrm{Re}\,h^*s + \sqrt{\left(\frac{h^*h}{2} - \left(\frac{h^*h}{2}\right)^2 + \mathrm{Re}\,h^*s\right)^2 - \sum_{i<j}|s_i\bar{h_j} - \bar{s_j}h_i|^2}}{h^*h} .$$

Proof. Let $H = hh^* - (h^*h/2)I$ and let $M_x = A - xI$ where x is a real parameter. To prove the theorem, we first slightly modify the idea behind the proof of Theorem 1 of Ref. [10] to get a lower bound of λ_1. and then apply (2.1). From a theorem of Ostrowski & Schneider found in Ref. [17] (which states that if $P_x = M_xH + HM_x^*$ is an Hermitian positive definite matrix then M_x and H have the same number of eigenvalues with positive real part, the same number of eigenvalues with negative real part, and no eigenvalue with zero real part), we determine an interval of x such that P_x is Hermitian positive definite for all x over that interval. Since $A+A^*=hh^*-I$ and $Ahh^*=sh^*$, a simple calculation yields

$$\begin{aligned} P_x =&M_xH+HM_x^* \\ =&xh^*hI+Ahh^*+hh^*A^*-\frac{h^*h}{2}(A+A^*) \\ =&\left(xh^*h+\frac{h^*h}{2}\right)I+sh^*+hs^*-\left(2x+\frac{h^*h}{2}\right)hh^*. \end{aligned}$$

Define $N_x=sh^*+hs^*-(2x+(h^*h/2))hh^*=sh^*+h(s^*-(2x+(h^*h/2))h^*)$. Since the rank of N_x is at most 2, we can readily derive its characteristic polynomial

$$\begin{aligned} |\lambda I-N_x| =&\lambda^n-\left(2\operatorname{Re}h^*s-\left(2x+\frac{h^*h}{2}\right)h^*h\right)\lambda^{n-1} \\ &+\sum_{i<j}\begin{vmatrix} 2\operatorname{Re}s_i\bar{h}_i-\left(2x+\frac{h^*h}{2}\right)h_i\bar{h}_i & s_i\bar{h}_j+h_i\bar{s}_j-\left(2x+\frac{h^*h}{2}\right)h_i\bar{h}_j \\ s_j\bar{h}_i+h_j\bar{s}_i-\left(2x+\frac{h^*h}{2}\right)h_j\bar{h}_i & 2\operatorname{Re}s_j\bar{h}_j-\left(2x+\frac{h^*h}{2}\right)h_j\bar{h}_j \end{vmatrix}\lambda^{n-2}. \end{aligned}$$

Splitting each of the above 2×2 determinants into four 2×2 determinants and then proceeding with a few simple manipulations so

$$\begin{aligned} &\begin{vmatrix} 2\operatorname{Re}s_i\bar{h}_i-\left(2x+\frac{h^*h}{2}\right)h_i\bar{h}_i & s_i\bar{h}_j+h_i\bar{s}_j-\left(2x+\frac{h^*h}{2}\right)h_i\bar{h}_j \\ s_j\bar{h}_i+h_j\bar{s}_i-\left(2x+\frac{h^*h}{2}\right)h_j\bar{h}_i & 2\operatorname{Re}s_j\bar{h}_j-\left(2x+\frac{h^*h}{2}\right)h_j\bar{h}_j \end{vmatrix} \\ =&\begin{vmatrix} 2\operatorname{Re}s_i\bar{h}_i & s_i\bar{h}_j+h_i\bar{s}_j \\ s_j\bar{h}_i+h_j\bar{s}_i & 2\operatorname{Re}s_j\bar{h}_j \end{vmatrix} \\ =&-|s_i\bar{h}_j-\bar{s}_jh_i|^2, \end{aligned}$$

we have that

$$|\lambda I-N_x|=\lambda^n-\left(2\operatorname{Re}h^*s-\left(2x+\frac{h^*h}{2}\right)h^*h\right)\lambda^{n-1}-\left(\sum_{i<j}|s_i\bar{h}_j-\bar{s}_jh_i|^2\right)\lambda^{n-2}.$$

Note that N_x has only two nonzero eigenvalues. From this polynomial, in order for P_x to be positive definite the following inequality must be true (the least eigenvalue of P_x must be positive):

$$\begin{aligned} &xh^*h+\frac{h^*h}{2}+\operatorname{Re}h^*s-\left(x+\frac{h^*h}{4}\right)h^*h-\sqrt{\left(\operatorname{Re}h^*s-\left(x+\frac{h^*h}{4}\right)h^*h\right)^2+\sum_{i<j}|s_i\bar{h}_j-\bar{s}_jh_i|^2} \\ =&\frac{h^*h}{2}-\left(\frac{h^*h}{2}\right)^2+\operatorname{Re}h^*s-\sqrt{\left(\operatorname{Re}h^*s-\left(x+\frac{h^*h}{4}\right)h^*h\right)^2+\sum_{i<j}|s_i\bar{h}_j-\bar{s}_jh_i|^2} \\ >&0\,. \end{aligned}$$

From the assumption (3.2), it follows that the above inequality is valid for x satisfying

$$x < \frac{-\left(\frac{h^*h}{2}\right)^2 + \operatorname{Re} h^*s + \sqrt{\left(\frac{h^*h}{2} - \left(\frac{h^*h}{2}\right)^2 + \operatorname{Re} h^*s\right)^2 - \sum_{i<j} |s_i\bar{h_j} - \bar{s_j}h_i|^2}}{h^*h},$$

and

$$x > \frac{-\left(\frac{h^*h}{2}\right)^2 + \operatorname{Re} h^*s - \sqrt{\left(\frac{h^*h}{2} - \left(\frac{h^*h}{2}\right)^2 + \operatorname{Re} h^*s\right)^2 - \sum_{i<j} |s_i\bar{h_j} - \bar{s_j}h_i|^2}}{h^*h}.$$

Since the eigenvalues of $H = hh^* - (h^*h/2)I$ are $h^*h/2$ and $-h^*h/2$ ($n-1$ times), by the Ostrowski-Schneider theorem stated above we see that, for any x that satisfies the above two inequalities, the matrix $A - xI$ has one real positive eigenvalue and $n-1$ eigenvalues with negative real part. This implies

$$\begin{aligned}\lambda_1 \geq &\frac{-\left(\frac{h^*h}{2}\right)^2 + \operatorname{Re} h^*s + \sqrt{\left(\frac{h^*h}{2} - \left(\frac{h^*h}{2}\right)^2 + \operatorname{Re} h^*s\right)^2 - \sum_{i<j} |s_i\bar{h_j} - \bar{s_j}h_i|^2}}{h^*h} \\ = &\frac{h^*h-2}{4} + \frac{-\frac{h^*h(h^*h-1)}{2} + \operatorname{Re} h^*s + \sqrt{\left(\frac{h^*h}{2} - \left(\frac{h^*h}{2}\right)^2 + \operatorname{Re} h^*s\right)^2 - \sum_{i<j} |s_i\bar{h_j} - \bar{s_j}h_i|^2}}{h^*h},\end{aligned}$$

and thus applying (2.1) to the above inequality $\forall\, j \geq 2$ we achieve

$$\sum_{i=2}^{j} \operatorname{Re} \lambda_i \leq \frac{3h^*h - 2j}{4} - \frac{\operatorname{Re} h^*s + \sqrt{\left(\frac{h^*h}{2} - \left(\frac{h^*h}{2}\right)^2 + \operatorname{Re} h^*s\right)^2 - \sum_{i<j} |s_i\bar{h_j} - \bar{s_j}h_i|^2}}{h^*h}.$$

Corollary 3.3. *Assume A is a 1-hypertournament matrix of order n with score vector $s = (s_1, s_2, \cdots, s_n)^t$ and $\sum_{i<j}(s_i - s_j)^2 < n^4/16$. Then A has exactly one real positive eigenvalue $\rho(A)$ such that*

$$\rho(A) \geq \frac{n-2}{4} + \frac{n}{4}\sqrt{1 - \frac{16}{n^4}\sum_{i<j}(s_i - s_j)^2},$$

and the real parts of other eigenvalues λ_i ($i > 1$) satisfy

$$\sum_{i=2}^{j} \operatorname{Re} \lambda_i \leq \frac{n-2j+2}{4} - \frac{n}{4}\sqrt{1 - \frac{16}{n^4}\sum_{i<j}(s_i - s_j)^2}, \quad 2 \leq j \leq n. \tag{3.3}$$

Proof. Since A is a 1-hypertournament matrix of order n, we have that $h^*h = n$, $\lambda_1 = \rho(A)$, and $h^*s = 1^tA1 = n(n-1)/2$. The proof follows by putting the parameters into the inequalities in Theorem 3.2. $\square$

Remark 3.2. When $j=2$ in (3.3), Corollary 3.3 is coincident with Theorem 1 of Ref. [10] — cf. the identity $(\sum_{i=1}^n s_i)^2 = n(\sum_{i=1}^n s_i^2) - \sum_{i<j}(s_i - s_j)^2$. However, Corollary 3.3 is more compact and convenient for application.

From Corollary 3.3, we see that if A is an $n \times n$ *1*-hypertournament matrix with score vector $s = (s_1, s_2, \cdots, s_n)^t$ and $\sum_{i<j}(s_i - s_j)^2 < n^2(n-1)/4$ then A has exactly one real positive eigenvalue $\rho(A)$ such that

$$\rho(A) > \frac{n-2}{4} + \frac{n}{4}\sqrt{1 - \frac{4(n-1)}{n^2}} = \frac{n-2}{2},$$

while $\sum_{i=2}^j \mathrm{Re}\,\lambda_j < -(j-2)/2$, $\forall j \geq 2$. Further, if $n > 3$ and A is nonnegative, then A is also primitive from Lemma 2.2. We thus have the following corollary.

Corollary 3.4. *Let A be an $n \times n$ 1-hypertournament matrix with score vector $s = (s_1, s_2, \cdots, s_n)^t$. If $\sum_{i<j}(s_i - s_j)^2 < n^2(n-1)/4$, then*

$$\rho(A) > \frac{n-2}{2}, \qquad \sum_{i=2}^{j} \mathrm{Re}\,\lambda_j < -\frac{j-2}{2}, \qquad \forall\, j \geq 2,$$

and therefore A has exactly one real positive eigenvalue.

It is notable that if $d_n = \max|s_i - s_j| \leq \sqrt{n/2}$ then

$$\sum_{i<j}(s_i - s_j)^2 < \frac{n(n-1)}{2} d_n^2 \leq \frac{n^2(n-1)}{4},$$

and from the identity $n^2(n-1)^2/4 = (\sum_{i=1}^n s_i)^2 = n(\sum_{i=1}^n s_i^2) - \sum_{i<j}(s_i - s_j)^2$ we also have

$$\frac{n^2(n-1)}{4} - \sum_{i=1}^{n} s_i^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} s_i^2 - \sum_{i<j}(s_i - s_j)^2\right).$$

From this relation it is easily checked that $\sum_{i=1}^n s_i^2 \leq n^2(n-1)/4$ if and only if $\sum_{i<j}(s_i - s_j)^2 \leq n^2(n-1)/4$. Consequently, the assertions of Corollaries 3.1, 3.2, Theorem 4, and Corollary 4.1 of Ref. [10] are also true if their (sharing) condition $\sum_{i=1}^n s_i^2 < n^2(n-1)/4$ is replaced with $\sum_{i<j}(s_i - s_j)^2 < n^2(n-1)/4$.

4. Concluding Remarks

We have derived new properties of h-pseudo-tournament matrices. The class of h-pseudo-tournament matrices is a generalisation of the well-studied tournament-like matrices, such as h-hypertournament matrices, generalised tournament matrices, tournament matrices, elliptic matrices, and Householder matrices. When the h-pseudo-tournament matrix discussed reduces to one of these tournament-like matrices, there are some new results, and several results provided elsewhere are proven as special cases. Further, the proofs of Theorems 2 and 3 of Ref. [11] generalise to h-pseudo-tournament matrices.

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