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Pseudo-Tournament Matrices and Their Eigenvalues

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Abstract. A tournament matrix and its corresponding directed graph both arise as a record of the outcomes of a round robin competition. An $n \times n$ complex matrix A is called h-pseudo-tournament if there exists a complex or real nonzero column vector h such that $A + A^* = hh^* - I$. This class of matrices is a generalisation of well-studied tournament-like matrices such as h-hypertournament matrices, generalised tournament matrices, tournament matrices, and elliptic matrices. We discuss the eigen-properties of an h-pseudo-tournament matrix, and obtain new results when the matrix specialises to one of these tournament-like matrices. Further, several results derived in previous articles prove to be corollaries of those reached here.

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1. Introduction

We let X^* and X^t represent the transpose conjugate and the transpose of a vector X, and use the same superscripts $*$ and t to likewise denote the transpose conjugate and transpose of a matrix. An $n \times n$ complex matrix A is called *h-pseudo-tournament* if there is a complex or real nonzero column vector h such that

$$A + A^* = hh^* - I. \tag{1.1}$$

This class of matrices was originally studied by Maybee & Pullman [13], and is a generalisation of the following classes of *tournament-like* matrices satisfying Eq. (1.1) that have received considerable attention in recent decades:

- if A is a real matrix with zero diagonal elements, then A is called an *h-hypertournament* matrix — in this case $h = (h_1, h_2, \cdots, h_n)^t$ where h_j is 1 or -1, $j = 1, \cdots, n$, and their spectral properties were derived [10, 13];

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- if A is a nonnegative matrix and $h = \mathit{1}$, where $\mathit{1}$ is the all ones column vector, then A is called a *generalised tournament* matrix [15,16];
- if A is a zero-one matrix (in this case $h = \mathit{1}$) then A is called a *tournament* matrix — cf. [1,5,6,12,18] and references therein.

Furthermore, if $-(A+A^*)$ is real then it is an *elliptic* matrix [4,19], and a real symmetric matrix is elliptic if it has exactly one positive eigenvalue; and $-(A+A^*)$ reduces to a *Householder* matrix if $h^*h = 2$ — cf. Ref. [9]. Incidentally, the techniques we use here are also applicable if the matrix A in Eq. (1.1) satisfies $A + A^* = -hh^* - I$. Without loss of generality, we assume throughout our discussion that h has no zero element.

A tournament matrix and its corresponding directed graph both arise as a record of the outcomes of a round robin competition. The need and desire to come up with player ranking schemes has motivated an extensive study of the combinatorial and spectral properties of tournament matrices and their generalisations. Hypertournament matrices, the generalised tournament matrices, and pseudo-tournament matrices can be understood as weighted tournaments. They not only provide a means for inquiring into the properties of more general tournaments but also are the source of matrix analytic challenges of independent interest, which interplay between matrix/graph theoretic and spectral properties. There is a wealth of literature that focuses on deriving algebraic or combinatorial attributes of these matrices [1,2,5,10,15,18]. In particular, Brauer & Gentry [1,2] showed that $-1/2 \le \mathrm{Re}\,\lambda \le (n-1)/2$ and $|\mathrm{Im}\,\lambda| \le \sqrt{n(n-1)/6}$ if λ is an eigenvalue of a tournament matrix A of order n. Moon & Pullman [16] then proved that similar results also hold for the generalised tournament matrices. Subsequently, Maybee & Pullman [13] considered the more general pseudo-tournament and h-hypertournament matrices, and proved the inequality $-1/2 \le \mathrm{Re}\,\lambda \le (n-1)/2$ for the h-hypertournament matrices. It is notable that any h-hypertournament matrix A is diagonally and orthogonally similar to a $\mathit{1}$-hypertournament matrix, because we then have $Dh = \mathit{1}$ where $h = (h_1, h_2, \cdots, h_n)^t$ and $D = diag(h_1, h_2, \cdots, h_n)$ with $h_i = 1$ or $-1\ \forall\, i$, such that $D^*(A+A^*)D = \mathit{1}\mathit{1}^t - I$). Accordingly, any investigation of the eigen-properties of an h-hypertournament matrix is equivalent to working on the eigen-properties of a $\mathit{1}$-hypertournament matrix.

If A is an $n \times n$ $\mathit{1}$-hypertournament matrix then $s = A\mathit{1}$ is called the *score vector* of A, and if $s = ((n-1)/2)\mathit{1}$ then A is said to be *regular*. The score vector s plays an important role for the eigenvalues of these matrices [10,13]. Any $\mathit{1}$-hypertournament matrix satisfies $s^t\mathit{1} = n(n-1)/2$ and $s^t s \ge n(n-1)^2/4$, with equality if and only if it is regular. Here we introduce similar definitions: for an $n \times n$ h-pseudo-tournament matrix A, we call $s = Ah$ the *pseudo-score vector* of A, and say A is *pseudo-regular* if $Ah = (h^*h - 1/2)h$. We note that a regular $\mathit{1}$-hypertournament matrix is a $\mathit{1}$-pseudo-regular tournament matrix; and also say that a $2n \times 2n$ $\mathit{1}$-hypertournament matrix T is *almost regular* if it has n row sums equal to $n-1$ and n row sums equal to n. These definitions will be used in our discussion on localising the eigenvalues of an h-pseudo-tournament matrix. We also use the following notation:

$\mathbf{C}^n(\mathbf{R}^n)$:	the n-dimensional complex (real) Euclidean vector space
$\lambda_i(A)$:	the ith eigenvalue of matrix A; sometimes, write $\lambda_i(A)$ simply as λ_i