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High Order Difference Schemes for a Time Fractional Differential Equation with Neumann Boundary Conditions

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Abstract. A compact finite difference scheme is derived for a time fractional differential equation subject to Neumann boundary conditions. The proposed scheme is second-order accurate in time and fourth-order accurate in space. In addition, a high order alternating direction implicit (ADI) scheme is also constructed for the two-dimensional case. The stability and convergence of the schemes are analysed using their matrix forms.

AMS subject classifications: 65M06, 65M12, 65M15, 35R11

Key words: Time fractional differential equation, Neumann boundary conditions, compact ADI scheme, weighted and shifted Grünwald difference operator, convergence.

1. Introduction

Fractional differential equations have become the focus of many studies, due to their various applications. Many problems from signal processing, anomalous diffusion and finance can be modelled more accurately using equations with fractional derivatives. For example, when studying universal electromagnetic responses involving the unification of diffusion and wave propagation phenomena, there are processes that are described by equations with time fractional derivatives of order between 1 and 2 (to “interpolate" between diffusion equations and wave equations). Reference may be made to the books [1,2] for further information, where theoretical results such as solution existence and uniqueness can also be found. One of the key features of fractional derivatives is nonlocal dependence, which causes difficulties when numerical schemes for solving fractional differential equations are designed, but substantial progress has been made in recent years – e.g. see Refs. [3–25], and in particular Refs. [21–25] where time fractional differential equations subject to Neumann boundary conditions are discussed.

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We consider high order finite difference schemes for the numerical solution in a region Ω for problems in the following form:

$$
\begin{aligned}
&{}_0^C D_t^{\gamma} u(\mathbf{x},t) = \kappa_1 \Delta u(\mathbf{x},t) - \kappa_2 u(\mathbf{x},t) + g(\mathbf{x},t)\,,\\
&\mathbf{x}\in\Omega\,, \qquad 0 < t \le T\,, \qquad 1<\gamma<2\,,
\end{aligned} \tag{1.1}
$$

subject to the initial conditions

$$
u(\mathbf{x},0) = \psi(\mathbf{x})\,, \qquad \frac{\partial u(\mathbf{x},0)}{\partial t} = \phi(\mathbf{x})\,, \qquad \mathbf{x}\in\bar{\Omega} = \Omega\cup\partial\Omega\,,
$$

and the zero flux boundary condition

$$
\frac{\partial u(\mathbf{x},t)}{\partial \mathbf{n}} = 0\,, \qquad \mathbf{x}\in\partial\Omega\,, \qquad 0<t\le T\,,
$$

where $\partial\Omega$ is the boundary of Ω, $\partial/\partial\mathbf{n}$ is the differentiation in the normal direction and κ_1, κ_2 are positive constants. Here ${}_0^C D_t^{\gamma} u$ denotes the Caputo fractional derivative of u with respect to the time variable t:

$$
{}_0^C D_t^{\gamma} u(\mathbf{x},t) = \frac{1}{\Gamma(2-\gamma)} \int_0^t \frac{\partial^2 u(\mathbf{x},s)}{\partial s^2} (t-s)^{1-\gamma} ds,
$$

where $\Gamma(\cdot)$ is the gamma function. Eq. (1.1) can be rewritten as (cf. [15])

$$
\begin{aligned}
\frac{\partial u(\mathbf{x},t)}{\partial t} = \phi(\mathbf{x}) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[\kappa_1 \frac{\partial^2 u(\mathbf{x},s)}{\partial x^2} - \kappa_2 u(\mathbf{x},s) \right] ds + f(\mathbf{x},t)\,,&\\
\mathbf{x}\in\Omega\,, \quad 0<t\le T\,,&
\end{aligned}
$$

where $0<\alpha=\gamma-1<1$, $f(\mathbf{x},t) = {}_0I_t^{\alpha} g(\mathbf{x},t)$, and ${}_0I_t^{\alpha}$ is the Riemann-Liouville fractional integral operator of order α defined as

$$
{}_0I_t^{\alpha} g(\mathbf{x},t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(\mathbf{x},s) ds\,.
$$

We also note that $\partial\psi(\mathbf{x})/\partial\mathbf{n} = 0$ for $\mathbf{x}\in\partial\Omega$.

By applying the weighted and shifted Grünwald difference (cf. [13, 14, 16]) to the Riemann-Liouville fractional integral, in this article we establish compact schemes with second-order temporal accuracy and fourth-order spatial accuracy. Our analysis is based on the matrix form of the schemes, which turns out to render some of the intuitive ideas on certain norms and inner products noted in previous related articles. In Section 2 and Section 3, we first consider the one-dimensional case of Eq. (1.1), where we propose a high order scheme and study its convergence. In Section 4, a high order alternating direction implicit scheme is then proposed for the two-dimensional case. Numerical examples are given in the last section.

2. Proposed Compact Difference Scheme

In this section, we develop a high order scheme for the one-dimensional case – viz.

$$
\begin{aligned}
&{}_0^C D_t^\gamma u(x,t) = \kappa_1 \frac{\partial^2 u(x,t)}{\partial x^2} - \kappa_2 u(x,t) + g(x,t)\,, \\
&\qquad 0 \le x \le L\,, \quad 0 < t \le T\,, \quad 1 < \gamma < 2\,, \qquad (2.1)\\
&u(x,0) = \psi(x)\,, \quad \frac{\partial u(x,0)}{\partial t} = \phi(x)\,, \quad 0 \le x \le L\,, \qquad (2.2)\\
&\frac{\partial u(0,t)}{\partial x} = 0\,, \quad \frac{\partial u(L,t)}{\partial x} = 0\,, \quad 0 < t \le T\,, \qquad (2.3)
\end{aligned}
$$

where we assume that $\psi \equiv 0$ in (2.2) without loss of generality (since we can solve the equation for $v(x,t) = u(x,t) - \psi(x)$ in general). An equivalent form of (2.1) is

$$
\frac{\partial u(x,t)}{\partial t} = \phi(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[\kappa_1 \frac{\partial^2 u(x,s)}{\partial x^2} - \kappa_2 u(x,s) \right] ds + f(x,t)\,, \qquad (2.4)
$$

where $0 \le x \le L$, $0 < t \le T$, $0 < \alpha = \gamma - 1 < 1$, $f(x,t) = {}_0I_t^\alpha g(x,t)$.

For given integers M and N, we discretize the equation using the spatial step size $h = L/M$ and the temporal step size $\tau = T/N$. For $i = 0,1,\cdots,M$ and $k = 0,1,\cdots,N$, we denote $x_i = ih$, $t_k = k\tau$ and $u^k = (u_0^k, u_1^k, \cdots, u_M^k)^T$, and then for the grid function $u = \{u_i^k \,|\, 0 \le i \le M,\ 0 \le k \le N\}$ approximating the solution we adopt the following:

$$
\delta_x u_{i-\frac12}^k = \frac{1}{h}(u_i^k - u_{i-1}^k)\,, \quad 1 \le i \le M\,,
$$

$$
\delta_x^2 u_i^k = \begin{cases} \frac{2}{h}\delta_x u_{\frac12}^k\,, & i = 0\,,\\ \frac{1}{h}(\delta_x u_{i+\frac12}^k - \delta_x u_{i-\frac12}^k)\,, & 1 \le i \le M-1\,,\\ -\frac{2}{h}\delta_x u_{M-\frac12}^k\,, & i = M\,, \end{cases}
$$

$$
\mathscr{H} u_i = \begin{cases} \frac16(5u_0 + u_1)\,, & i = 0\,,\\ \frac{1}{12}(u_{i-1} + 10u_i + u_{i+1})\,, & 1 \le i \le M-1\,,\\ \frac16(u_{M-1} + 5u_M)\,, & i = M\,, \end{cases}
$$

$$
\langle u, v\rangle = h\sum_{i=0}^M u_i v_i\,, \quad \|u\|^2 = \langle u,u\rangle\,, \quad \|u\|_\infty = \max_{0\le i\le M} |u_i|\,.
$$

The discretisation of $\partial^2 u/\partial x^2$ is based on the following lemma.

Lemma 2.1. (cf. Ref. [25]). *Denote* $\zeta(s) = (1-s)^3[5 - 3(1-s)^2]$.

(I) *If* $f(x) \in \mathscr{C}^6[x_0, x_1]$, *then*

$$
\begin{aligned}
&\left[\frac56 f''(x_0) + \frac16 f''(x_1)\right] - \frac{2}{h}\left[\frac{f(x_1) - f(x_0)}{h} - f'(x_0)\right] \\
&= -\frac{h}{6} f'''(x_0) + \frac{h^3}{90} f^{(5)}(x_0) + \frac{h^4}{180}\int_0^1 f^{(6)}(x_0 + sh)\zeta(s)ds\,.
\end{aligned}
$$

(II) *If* $f(x)\in\mathscr{C}^6[x_{M-1},x_M]$, *then*

$$\left[\frac{1}{6}f''(x_{M-1})+\frac{5}{6}f''(x_M)\right]-\frac{2}{h}\left[f'(x_M)-\frac{f(x_M)-f(x_{M-1})}{h}\right]$$
$$=\frac{h}{6}f'''(x_M)-\frac{h^3}{90}f^{(5)}(x_M)+\frac{h^4}{180}\int_0^1 f^{(6)}(x_M-sh)\zeta(s)ds\,.$$

(III) *If* $f(x)\in\mathscr{C}^6[x_{i-1},x_{i+1}],\ 1\le i\le M-1$, *then*

$$\frac{1}{12}\left[f''(x_{i-1})+10f''(x_i)+f''(x_{i+1})\right]-\frac{1}{h^2}\left[f(x_{i-1})-2f(x_i)+f(x_{i+1})\right]$$
$$=\frac{h^4}{360}\int_0^1\left[f^{(6)}(x_i-sh)+f^{(6)}(x_i+sh)\right]\zeta(s)ds\,.$$

Our numerical scheme for Eq. (2.1) is derived using the equivalent form Eq. (2.4). We introduce the shifted Grünwald difference to the Riemann-Liouville fractional integral

$$\mathscr{A}^{\alpha}_{\tau,r}f(t)=\tau^{\alpha}\sum_{k=0}^{\infty}\omega_k f(t-(k-r)\tau)$$

where $\omega_k=(-1)^k\binom{-\alpha}{k}$, to obtain the second-order approximation for Riemann-Liouville fractional integrals as follows [13,16].

Lemma 2.2. *Let* $f(t)$, ${}_{-\infty}I_t^{2-\alpha}f$ *and* $(i\omega)^{2-\alpha}\mathscr{F}[f](\omega)$ *belong to* $L^1(\mathbb{R})$*, and define the weighted and shifted difference operator by*

$$\mathscr{I}^{\alpha}_{\tau,p,q}f(t)=\frac{2q+\alpha}{2(q-p)}\mathscr{A}^{\alpha}_{\tau,p}f(t)+\frac{2p+\alpha}{2(p-q)}\mathscr{A}^{\alpha}_{\tau,q}f(t)\,.$$

Then we have

$$\mathscr{I}^{\alpha}_{\tau,p,q}f(t)={}_{-\infty}I_t^{\alpha}f(t)+O(\tau^2)$$

for $t\in\mathbb{R}$*, where* p *and* q *are integers and* $p\neq q$*.*

With $(p,q)=(0,-1)$, which yields $\frac{2q+\alpha}{2(q-p)}=1-\frac{\alpha}{2}$ and $\frac{2p+\alpha}{2(p-q)}=\frac{\alpha}{2}$ in Lemma 2.2, we get

$$\begin{aligned}{}_0I_t^{\alpha}u(x_i,t_{n+1})&=\tau^{\alpha}\left[\left(1-\frac{\alpha}{2}\right)\sum_{k=0}^{n+1}\omega_k u_i^{n+1-k}+\frac{\alpha}{2}\sum_{k=0}^{n}\omega_k u_i^{n-k}\right]+O(\tau^2)\\&=\tau^{\alpha}\sum_{k=0}^{n+1}\lambda_k u_i^{n+1-k}+O(\tau^2)\,,\end{aligned}$$

and

$$\begin{aligned}{}_0I_t^{\alpha}u_{xx}(x_i,t_{n+1})&=\tau^{\alpha}\left[\left(1-\frac{\alpha}{2}\right)\sum_{k=0}^{n+1}\omega_k\delta_x^2 u_i^{n+1-k}+\frac{\alpha}{2}\sum_{k=0}^{n}\omega_k\delta_x^2 u_i^{n-k}\right]+O(\tau^2+h^2)\\&=\tau^{\alpha}\sum_{k=0}^{n+1}\lambda_k\delta_x^2 u_i^{n+1-k}+O(\tau^2+h^2)\,,\end{aligned}$$

where

$$\lambda_0=(1-\frac{\alpha}{2})\omega_0\,,\qquad \lambda_k=(1-\frac{\alpha}{2})\omega_k+\frac{\alpha}{2}\omega_{k-1}\,,\qquad k\geq 1\,. \tag{2.5}$$

A weighted Crank-Nicolson scheme for Eq. (2.4) is therefore

$$\begin{aligned}\frac{u_i^{n+1}-u_i^n}{\tau}=&\phi_i+\frac{\tau^\alpha}{2}\Big[\sum_{k=0}^{n+1}\lambda_k(\kappa_1\delta_x^2u_i^{n+1-k}-\kappa_2u_i^{n+1-k})+\sum_{k=0}^{n}\lambda_k(\kappa_1\delta_x^2u_i^{n-k}-\kappa_2u_i^{n-k})\Big]\\&+\frac{1}{2}(f_i^n+f_i^{n+1}).\end{aligned}$$

To derive a higher order scheme, we follow Ref. [25]. Thus beginning with $i=0$, one has

$$\begin{aligned}&\mathscr{H}(u_0^{n+1}-u_0^n)\\=&\frac{\tau^{\alpha+1}}{2}\sum_{k=0}^{n+1}\lambda_k\bigg(\kappa_1\Big[\frac{2}{h}\delta_xu_{\frac{1}{2}}^{n+1-k}-\frac{2}{h}\frac{\partial u(0,t_{n+1-k})}{\partial x}-\frac{h}{6}\frac{\partial^3u(0,t_{n+1-k})}{\partial x^3}+\frac{h^3}{90}\frac{\partial^5u(0,t_{n+1-k})}{\partial x^5}\Big]\\&-\kappa_2\mathscr{H}u_0^{n+1-k}\bigg)\\&+\frac{\tau^{\alpha+1}}{2}\sum_{k=0}^{n}\lambda_k\bigg(\kappa_1\Big[\frac{2}{h}\delta_xu_{\frac{1}{2}}^{n-k}-\frac{2}{h}\frac{\partial u(0,t_{n-k})}{\partial x}-\frac{h}{6}\frac{\partial^3u(0,t_{n-k})}{\partial x^3}+\frac{h^3}{90}\frac{\partial^5u(0,t_{n-k})}{\partial x^5}\Big]\\&-\kappa_2\mathscr{H}u_0^{n-k}\bigg)+\tau\mathscr{H}\phi_0+\frac{\tau}{2}\mathscr{H}(f_0^n+f_0^{n+1})+\tau R_0^{n+1},\end{aligned} \tag{2.6}$$

where $R_0^{n+1}=O(\tau^2+h^4)$. We can now differentiate Eq. (2.1) with respect to x to give

$${}_0^CD_t^\gamma\frac{\partial u(x,t)}{\partial x}=\kappa_1\frac{\partial^3u(x,t)}{\partial x^3}-\kappa_2\frac{\partial u(x,t)}{\partial x}+g_x(x,t).$$

Letting $x\to 0^+$ and noting the boundary condition (2.3), we have

$$\kappa_1\frac{\partial^3u(0,t)}{\partial x^3}=-g_x(0,t)\,. \tag{2.7}$$

With the Caputo fractional derivative operator ${}_0^CD_t^\gamma$ acting on Eq. (2.7), it follows that

$${}_0^CD_t^\gamma\frac{\partial^3u(0,t)}{\partial x^3}=-\frac{1}{\kappa_1}{}_0^CD_t^\gamma g_x(0,t)\,. \tag{2.8}$$

Meanwhile, differentiating equation (2.1) three times with respect to x yields

$${}_0^CD_t^\gamma\frac{\partial^3u(x,t)}{\partial x^3}=\kappa_1\frac{\partial^5u(x,t)}{\partial x^5}-\kappa_2\frac{\partial^3u(x,t)}{\partial x^3}+g_{xxx}(x,t)\,. \tag{2.9}$$

Once again, we let $x\to 0^+$ in (2.9). We can then substitute Eqs. (2.7) and (2.8) to Eq. (2.9) to obtain

$$\kappa_1\frac{\partial^5u(0,t)}{\partial x^5}=-g_{xxx}(0,t)-\frac{\kappa_2}{\kappa_1}g_x(0,t)-\frac{1}{\kappa_1}{}_0^CD_t^\gamma g_x(0,t)\,. \tag{2.10}$$

Inserting Eqs. (2.7) and (2.10) into Eq. (2.6) and noting the boundary condition (2.3), on omitting small terms the compact scheme for $i = 0$ becomes

$$\begin{aligned}
&\mathscr{H}(u_0^{n+1} - u_0^n)\\
=&\frac{\tau^{\alpha+1}}{2}\left[\sum_{k=0}^{n+1}\lambda_k(\kappa_1\delta_x^2 u_0^{n+1-k} - \kappa_2\mathscr{H}u_0^{n+1-k}) + \sum_{k=0}^{n}\lambda_k(\kappa_1\delta_x^2 u_0^{n-k} - \kappa_2\mathscr{H}u_0^{n-k})\right]\\
&+\frac{\tau^{\alpha+1}}{2}\sum_{k=0}^{n+1}\lambda_k\left(\frac{h}{6}(g_x)_0^{n+1-k} - \frac{h^3}{90}\left[(g_{xxx})_0^{n+1-k} + \frac{\kappa_2}{\kappa_1}(g_x)_0^{n+1-k} + \frac{1}{\kappa_1}({}_0^C D_t^{\alpha+1} g_x)_0^{n+1-k}\right]\right)\\
&+\frac{\tau^{\alpha+1}}{2}\sum_{k=0}^{n}\lambda_k\left(\frac{h}{6}(g_x)_0^{n-k} - \frac{h^3}{90}\left[(g_{xxx})_0^{n-k} + \frac{\kappa_2}{\kappa_1}(g_x)_0^{n-k} + \frac{1}{\kappa_1}({}_0^C D_t^{\alpha+1} g_x)_0^{n-k}\right]\right)\\
&+\tau\mathscr{H}\phi_0 + \frac{\tau}{2}\mathscr{H}(f_0^n + f_0^{n+1}), \qquad 0 \le n \le N-1 .
\end{aligned} \tag{2.11}$$

The scheme at the other end can similarly be derived as

$$\begin{aligned}
&\mathscr{H}(u_M^{n+1} - u_M^n)\\
=&\frac{\tau^{\alpha+1}}{2}\left[\sum_{k=0}^{n+1}\lambda_k(\kappa_1\delta_x^2 u_M^{n+1-k} - \kappa_2\mathscr{H}u_M^{n+1-k}) + \sum_{k=0}^{n}\lambda_k(\kappa_1\delta_x^2 u_M^{n-k} - \kappa_2\mathscr{H}u_M^{n-k})\right]\\
&-\frac{\tau^{\alpha+1}}{2}\sum_{k=0}^{n+1}\lambda_k\left(\frac{h}{6}(g_x)_M^{n+1-k} - \frac{h^3}{90}\left[(g_{xxx})_M^{n+1-k} + \frac{\kappa_2}{\kappa_1}(g_x)_M^{n+1-k} + \frac{1}{\kappa_1}({}_0^C D_t^{\alpha+1} g_x)_M^{n+1-k}\right]\right)\\
&-\frac{\tau^{\alpha+1}}{2}\sum_{k=0}^{n}\lambda_k\left(\frac{h}{6}(g_x)_M^{n-k} - \frac{h^3}{90}\left[(g_{xxx})_M^{n-k} + \frac{\kappa_2}{\kappa_1}(g_x)_M^{n-k} + \frac{1}{\kappa_1}({}_0^C D_t^{\alpha+1} g_x)_M^{n-k}\right]\right)\\
&+\tau\mathscr{H}\phi_M + \frac{\tau}{2}\mathscr{H}(f_M^n + f_M^{n+1}), \qquad 0 \le n \le N-1 .
\end{aligned} \tag{2.12}$$

At any internal grid point, the scheme can be written

$$\begin{aligned}
&\mathscr{H}(u_i^{n+1} - u_i^n)\\
=&\frac{\tau^{\alpha+1}}{2}\left[\sum_{k=0}^{n+1}\lambda_k(\kappa_1\delta_x^2 u_i^{n+1-k} - \kappa_2\mathscr{H}u_i^{n+1-k}) + \sum_{k=0}^{n}\lambda_k(\kappa_1\delta_x^2 u_i^{n-k} - \kappa_2\mathscr{H}u_i^{n-k})\right]\\
&+\tau\mathscr{H}\phi_i + \frac{\tau}{2}\mathscr{H}(f_i^n + f_i^{n+1}), \quad 1 \le i \le M-1, \quad 0 \le n \le N-1 .
\end{aligned} \tag{2.13}$$

The approximate solution is then found assuming

$$u_i^0 = 0, \qquad 0 \le i \le M . \tag{2.14}$$

At each time level, the difference scheme Eqs. (2.11)–(2.14) is a linear tridiagonal system with a strictly diagonal dominant coefficient matrix, so it has a unique solution.

3. Stability and Convergence of the Compact Scheme

The main convergence result for the scheme outlined in the previous section can be established via the following lemmas.

Lemma 3.1. (cf. Ref. [16]) *With* $\{\lambda_n\}_{n=0}^{\infty}$ *defined as in* (2.5)*, for any positive integer* k *and real vector* $(v_1, v_2, \cdots, v_k)^T \in \mathbb{R}^k$ *we have*

$$\sum_{n=0}^{k-1}\left(\sum_{p=0}^{n}\lambda_p v_{n+1-p}\right)v_{n+1} \geq 0\,.$$

Lemma 3.2. ([26]) *Assume that* $\{k_n\}$ *and* $\{p_n\}$ *are nonnegative sequences, and the sequence* $\{\phi_n\}$ *satisfies*

$$\phi_0 \leq g_0\,, \qquad \phi_n \leq g_0 + \sum_{l=0}^{n-1} p_l + \sum_{l=0}^{n-1} k_l \phi_l\,, \qquad n \geq 1\,,$$

where $g_0 \geq 0$*. Then the sequence* $\{\phi_n\}$ *satisfies*

$$\phi_n \leq \left(g_0 + \sum_{l=0}^{n-1} p_l\right)\exp\left(\sum_{l=0}^{n-1} k_l\right), \qquad n \geq 1\,.$$

Our compact difference scheme (2.11)–(2.14) thus has high order convergence, as follows.

Theorem 3.1. *Assume* $u(x,t) \in \mathscr{C}_{x,t}^{6,2}([0,L]\times[0,T])$ *is the solution of the problem* (2.1)–(2.3) *and* $\{u_i^k | 0 \leq i \leq M,\ 0 \leq k \leq N\}$ *is a solution of the finite difference scheme* (2.11)–(2.14)*, respectively. Denote*

$$e_i^k = u(x_i,t_k) - u_i^k\,, \qquad 0 \leq i \leq M\,, \qquad 0 \leq k \leq N\,.$$

Then there exists a positive constant c *such that*

$$\|e^k\| \leq c(\tau^2 + h^4)\,, \qquad 0 \leq k \leq N\,.$$

Proof. We can easily get the following error equation:

$$\begin{aligned}\bar{C}(e^{k+1} - e^k) = &-\frac{\kappa_1\tau^{\alpha+1}}{2h^2}\sum_{l=0}^{k}\lambda_l\bar{Q}(e^{k+1-l} + e^{k-l}) \\ &-\frac{\kappa_2\tau^{\alpha+1}}{2}\sum_{l=0}^{k}\lambda_l\bar{C}(e^{k+1-l} + e^{k-l}) + \tau\bar{R}^{k+1}\,, \quad e_i^0 = 0\,, \quad 0 \leq i \leq M\,, \end{aligned} \tag{3.1}$$

where $\|\bar{R}^{k+1}\| \leq c_1(\tau^2 + h^4)$,

$$\bar{C} = \frac{1}{12}\begin{pmatrix} 10 & 2 & & & \\ 1 & 10 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 10 & 1 \\ & & & 2 & 10 \end{pmatrix}, \quad \bar{Q} = \begin{pmatrix} 2 & -2 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}. \tag{3.2}$$

Multiplying Eq. (3.1) by $\frac{1}{2} \oplus I \oplus \frac{1}{2}$ where I is the identity matrix, we get

$$\begin{aligned} C(e^{k+1}-e^k) = &-\frac{\kappa_1 \tau^{\alpha+1}}{2h^2}\sum_{l=0}^{k}\lambda_l Q(e^{k+1-l}+e^{k-l}) - \frac{\kappa_2 \tau^{\alpha+1}}{2}\sum_{l=0}^{k}\lambda_l C(e^{k+1-l}+e^{k-l}) \\ &+ \tau R^{k+1}, \end{aligned} \tag{3.3}$$

where $\|R^{k+1}\| \le c_2(\tau^2+h^4)$,

$$C = \frac{1}{12}\begin{pmatrix} 5 & 1 & & & \\ 1 & 10 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 10 & 1 \\ & & & 1 & 5 \end{pmatrix} = E^2 \tag{3.4}$$

(thus E the square root of C), and

$$Q = \begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix} = S^T S \tag{3.5}$$

with

$$S = \begin{pmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{pmatrix} \in \mathbb{R}^{M\times M+1}. \tag{3.6}$$

Here we have used the fact that $C = \frac{1}{12}tri[1,5,1] + \frac{5}{12}(0 \oplus I \oplus 0)$ is positive definite. Consequently, on multiplying Eq. (3.3) by $h(e^{k+1}+e^k)^T$, we obtain

$$\begin{aligned} &h(e^{k+1}+e^k)^T C(e^{k+1}-e^k) \\ = &-\frac{\kappa_1 \tau^{\alpha+1}}{2h}\sum_{l=0}^{k}\lambda_l (e^{k+1}+e^k)^T S^T S(e^{k+1-l}+e^{k-l}) \\ &-\frac{\kappa_2 h \tau^{\alpha+1}}{2}\sum_{l=0}^{k}\lambda_l (e^{k+1}+e^k)^T E^2(e^{k+1-l}+e^{k-l}) + \tau h(e^{k+1}+e^k)^T R^{k+1}. \end{aligned}$$

Summing up for $0 \le k \le n-1$ and noting that

$$\begin{aligned} &h(e^{k+1}+e^k)^T C(e^{k+1}-e^k) = h\left[(e^{k+1})^T C e^{k+1} - (e^k)^T C e^k\right], \\ &h(e^n)^T C e^n \ge \frac{1}{4}\|e^n\|^2, \end{aligned}$$

from Lemma 3.1 we then have

$$
\begin{aligned}
&\frac{1}{4}\|e^n\|^2\\
\le&\tau h(e^n+e^{n-1})^T R^n+\tau h\sum_{k=0}^{n-2}(e^{k+1}+e^k)^T R^{k+1}\\
\le&\frac{1}{5}\|e^n\|^2+\frac{5\tau^2}{4}\|R^n\|^2+\frac{\tau}{2}\|e^{n-1}\|^2+\frac{\tau}{2}\|R^n\|^2+\frac{\tau}{2}\sum_{k=1}^{n-1}\|e^k\|^2+\frac{\tau}{2}\sum_{k=1}^{n-2}\|e^k\|^2+\tau\sum_{k=1}^{n-1}\|R^k\|^2\\
\le&\frac{1}{5}\|e^n\|^2+\frac{5\tau^2}{4}\|R^n\|^2+\tau\sum_{k=1}^{n-1}\|e^k\|^2+\tau\sum_{k=1}^{n}\|R^k\|^2\,,
\end{aligned}
$$

whence

$$
\begin{aligned}
\|e^n\|^2\le&25\tau^2\|R^n\|^2+20\tau\sum_{k=1}^{n-1}\|e^k\|^2+20\tau\sum_{k=1}^{n}\|R^k\|^2\\
\le&20\tau\sum_{k=1}^{n-1}\|e^k\|^2+c_3(\tau^2+h^4)^2\,,
\end{aligned}
$$

so the desired result follows from Lemma 3.2. □

Remark 3.1. One can adopt aspects in the proof for Theorem 3.1 to show that the proposed compact scheme (2.11)–(2.14) is unconditionally stable. Indeed, consider the solution $\{v_i^k\}$ of

$$
\begin{aligned}
&\mathscr{H}(v_0^{k+1}-v_0^k)\\
=&\frac{\tau^{\alpha+1}}{2}\left[\sum_{l=0}^{k+1}\lambda_l(\kappa_1\delta_x^2 v_0^{k+1-l}-\kappa_2\mathscr{H}v_0^{k+1-l})+\sum_{l=0}^{k}\lambda_l(\kappa_1\delta_x^2 v_0^{k-l}-\kappa_2\mathscr{H}v_0^{k-l})\right]\\
&+\frac{\tau^{\alpha+1}}{2}\sum_{l=0}^{k+1}\lambda_l\left(\frac{h}{6}(g_x)_0^{k+1-l}-\frac{h^3}{90}\left[(g_{xxx})_0^{k+1-l}+\frac{\kappa_2}{\kappa_1}(g_x)_0^{k+1-l}+\frac{1}{\kappa_1}({}_0^C D_t^{\alpha+1}g_x)_0^{k+1-l}\right]\right)\\
&+\frac{\tau^{\alpha+1}}{2}\sum_{l=0}^{k}\lambda_l\left(\frac{h}{6}(g_x)_0^{k-l}-\frac{h^3}{90}\left[(g_{xxx})_0^{k-l}+\frac{\kappa_2}{\kappa_1}(g_x)_0^{k-l}+\frac{1}{\kappa_1}({}_0^C D_t^{\alpha+1}g_x)_0^{k-l}\right]\right)\\
&+\tau\mathscr{H}(\phi_0+\tilde{\rho}_0)+\frac{\tau}{2}\mathscr{H}(f_0^k+f_0^{k+1})\,,\qquad 0\le k\le N-1\,,
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
&\mathscr{H}(v_M^{k+1}-v_M^k)\\
=&\frac{\tau^{\alpha+1}}{2}\left[\sum_{l=0}^{k+1}\lambda_l(\kappa_1\delta_x^2 v_M^{k+1-l}-\kappa_2\mathscr{H}v_M^{k+1-l})+\sum_{l=0}^{k}\lambda_l(\kappa_1\delta_x^2 v_M^{k-l}-\kappa_2\mathscr{H}v_M^{k-l})\right]\\
&-\frac{\tau^{\alpha+1}}{2}\sum_{l=0}^{k+1}\lambda_l\left(\frac{h}{6}(g_x)_M^{k+1-l}-\frac{h^3}{90}\left[(g_{xxx})_M^{k+1-l}+\frac{\kappa_2}{\kappa_1}(g_x)_M^{k+1-l}+\frac{1}{\kappa_1}({}_0^C D_t^{\alpha+1}g_x)_M^{k+1-l}\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\tau^{\alpha+1}}{2}\sum_{l=0}^{k}\lambda_l\left(\frac{h}{6}(g_x)_M^{k-l}-\frac{h^3}{90}\left[(g_{xxx})_M^{k-l}+\frac{\kappa_2}{\kappa_1}(g_x)_M^{k-l}+\frac{1}{\kappa_1}({}_0^CD_t^{\alpha+1}g_x)_M^{k-l}\right]\right)\\
&+\tau\mathscr{H}(\phi_M+\tilde{\rho}_M)+\frac{\tau}{2}\mathscr{H}(f_M^k+f_M^{k+1})\,,\qquad 0\le k\le N-1\,, \qquad (3.8)\\
&\mathscr{H}(v_i^{k+1}-v_i^k)\\
=&\frac{\tau^{\alpha+1}}{2}\left[\sum_{l=0}^{k+1}\lambda_l(\kappa_1\delta_x^2v_i^{k+1-l}-\kappa_2\mathscr{H}v_i^{k+1-l})+\sum_{l=0}^{k}\lambda_l(\kappa_1\delta_x^2v_i^{k-l}-\kappa_2\mathscr{H}v_i^{k-l})\right]\\
&+\tau\mathscr{H}(\phi_i+\tilde{\rho}_i)+\frac{\tau}{2}\mathscr{H}(f_i^k+f_i^{k+1})\,,\qquad 1\le i\le M-1,\quad 0\le k\le N-1\,, \qquad (3.9)
\end{aligned}
$$

with $v_i^0=\rho_i$, $0\le i\le M$. Thus from (2.11)–(2.13) and (3.7)–(3.9), one can check that $\varepsilon_i^l=v_i^l-u_i^l-\rho_i$ satisfy

$$
\begin{aligned}
&\mathscr{H}(\varepsilon_i^{k+1}-\varepsilon_i^k)\\
=&\frac{\tau^{\alpha+1}}{2}\left[\sum_{l=0}^{k+1}\lambda_l(\kappa_1\delta_x^2\varepsilon_i^{k+1-l}-\kappa_2\mathscr{H}\varepsilon_i^{k+1-l})+\sum_{l=0}^{k}\lambda_l(\kappa_1\delta_x^2\varepsilon_i^{k-l}-\kappa_2\mathscr{H}\varepsilon_i^{k-l})\right]+\tau\mathscr{H}\tilde{\rho}_i\\
&+\frac{\tau^{\alpha+1}}{2}\left[\sum_{l=0}^{k+1}\lambda_l(\kappa_1\delta_x^2\rho_i-\kappa_2\mathscr{H}\rho_i)+\sum_{l=0}^{k}\lambda_l(\kappa_1\delta_x^2\rho_i-\kappa_2\mathscr{H}\rho_i)\right]\,,\\
&\quad 0\le i\le M\,,\ \ 0\le k\le N-1\,,\\
&\varepsilon_i^0=0\,,\qquad 0\le i\le M\,. \qquad (3.10)
\end{aligned}
$$

By following the proof for Theorem 3.1 and noting $\tau^\alpha\sum\limits_{l=0}^{k+1}\lambda_l=\frac{1}{\Gamma(\alpha+1)}+O(\tau)$, we then have the estimate

$$
\begin{aligned}
\|\varepsilon^k\|^2&\le 20\tau\sum_{l=0}^{k-1}\|\varepsilon^l\|^2+\left[\frac{5}{\Gamma(\alpha+1)}+1\right]^2\left[\|\kappa_1\delta_x^2\rho\|^2+\|\kappa_2\rho\|^2+\|\tilde{\rho}\|^2\right]\\
&\le e^{20T}\left[\frac{5}{\Gamma(\alpha+1)}+1\right]^2\left[\|\kappa_1\delta_x^2\rho\|^2+\|\kappa_2\rho\|^2+\|\tilde{\rho}\|^2\right]\,. \qquad (3.11)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|v^k-u^k\|&\le\|v^k-u^k-\rho\|+\|\rho\|\\
&\le e^{10T}\left[\frac{5}{\Gamma(\alpha+1)}+1\right]\sqrt{\|\kappa_1\delta_x^2\rho\|^2+\|\kappa_2\rho\|^2+\|\tilde{\rho}\|^2}+\|\rho\|\,,
\end{aligned}
$$

and hence the stability of the scheme.

4. The Compact ADI Scheme for the Two-Dimensional Problem

In this section, we consider the two-dimensional case

$$ {}_0^C D_t^\gamma u = \Delta u - u + g(x,y,t)\,, \quad (x,y)\in\Omega\,, \quad 0<t\le T\,, \quad 1<\gamma<2\,, \tag{4.1} $$

$$ u(x,y,0)=0\,, \quad \frac{\partial u(x,y,0)}{\partial t} = \phi(x,y)\,, \quad (x,y)\in\bar{\Omega}=\Omega\cup\partial\Omega\,, \tag{4.2} $$

$$ \frac{\partial u(x,y,t)}{\partial n}\Big|_{\partial\Omega} = 0\,, \quad (x,y)\in\partial\Omega\,, \quad 0<t\le T\,, \tag{4.3} $$

where Δ is the two-dimensional Laplacian and n is the unit outward normal vector of the domain $\Omega=(0,L_1)\times(0,L_2)$ with boundary $\partial\Omega$. An equivalent form of Eq. (4.1) is

$$ \frac{\partial u(x,y,t)}{\partial t} = \phi(x,y) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\Delta u(x,y,s)-u(x,y,s)]ds + f(x,y,t)\,, \tag{4.4} $$

where $(x,y)\in\Omega,\ 0<t\le T,\ 0<\alpha=\gamma-1<1,\ f(x,y,t)={}_0I_t^\alpha g(x,y,t)$. The discretisation of Eq. (4.4) is carried out in steps similar to that of the one-dimensional case. Thus we let $h_1=L_1/M_1,\ h_2=L_2/M_2$ and $\tau=T/N$ be the respective spatial and temporal step sizes, where $M_1,\ M_2$ and N are some given integers. For $i=0,1,\cdots,M_1,\ j=0,1,\cdots,M_2$ and $k=0,1,\cdots,N$, we denote $x_i=ih_1,\ y_j=jh_2,\ t_k=k\tau$ and introduce the following notation on a grid function $u=\{u_{ij}^k|0\le i\le M_1,\ 0\le j\le M_2,\ 0\le k\le N\}$:

$$ \delta_x u_{i-\frac12,j} = \frac{1}{h_1}(u_{ij}-u_{i-1,j})\,, $$

$$ \delta_x^2 u_{ij} = \begin{cases} \frac{2}{h_1}\delta_x u_{\frac12,j}\,, & i=0,\ 0\le j\le M_2\,, \\ \frac{1}{h_1}(\delta_x u_{i+\frac12,j}-\delta_x u_{i-\frac12,j})\,, & 1\le i\le M_1-1,\ 0\le j\le M_2\,, \\ -\frac{2}{h_1}\delta_x u_{M_1-\frac12,j}\,, & i=M_1,\ 0\le j\le M_2\,, \end{cases} $$

$$ \mathscr{H}_x u_{ij} = \begin{cases} \frac16(5u_{0,j}+u_{1,j})\,, & i=0\,,\ 0\le j\le M_2\,, \\ \frac{1}{12}(u_{i-1,j}+10u_{i,j}+u_{i+1,j})\,, & 1\le i\le M_1-1\,,\ 0\le j\le M_2\,, \\ \frac16(u_{M_1-1,j}+5u_{M_1,j})\,, & i=M_1\,,\ 0\le j\le M_2\,. \end{cases} $$

One can define similar notation in the y direction. We further denote

$$ \mathscr{H}u_{ij} = \mathscr{H}_x\mathscr{H}_y u_{ij}\,, \qquad \Lambda u_{ij} = (\mathscr{H}_y\delta_x^2+\mathscr{H}_x\delta_y^2)u_{ij}\,, $$

$$ \langle u,v\rangle = h_1h_2\sum_{i=0}^{M_1}\sum_{j=0}^{M_2} u_{ij}v_{ij}\,, \qquad \|u\|^2=\langle u,u\rangle\,, \qquad \|u\|_\infty = \max_{0\le i\le M_1,\ 0\le j\le M_2}|u_{ij}|\,. $$

With the above preparation, we obtain the compact ADI scheme as follows. We denote $\mu=\tau^{\alpha+1}/2$ and $G_{ij}^n=(G_{ij}^1)^n+(G_{ij}^2)^n$, where

$$
(G_{ij}^1)^n=\begin{cases}
\mu\sum\limits_{k=0}^{n}\lambda_k\bigg(\dfrac{h_1}{6}\mathscr{H}_y(g_x)_{0,j}^{n-k}-\dfrac{h_1^3}{90}\mathscr{H}_y\Big[(g_{xxx})_{0,j}^{n-k}-(g_{xyy})_{0,j}^{n-k}+(g_x)_{0,j}^{n-k}\\
\qquad+({}_0^CD_t^{\alpha+1}g_x)_{0,j}^{n-k}\Big]\bigg), \quad i=0\,, \quad 0\le j\le M_2\,,\\
0\,, \quad 1\le i\le M_1-1\,, \quad 0\le j\le M_2\,,\\
-\mu\sum\limits_{k=0}^{n}\lambda_k\bigg(\dfrac{h_1}{6}\mathscr{H}_y(g_x)_{M_1,j}^{n-k}-\dfrac{h_1^3}{90}\mathscr{H}_y\Big[(g_{xxx})_{M_1,j}^{n-k}-(g_{xyy})_{M_1,j}^{n-k}+(g_x)_{M_1,j}^{n-k}\\
\qquad+({}_0^CD_t^{\alpha+1}g_x)_{M_1,j}^{n-k}\Big]\bigg), \quad i=M_1\,, \quad 0\le j\le M_2\,,
\end{cases}
$$

$$
(G_{ij}^2)^n=\begin{cases}
\mu\sum\limits_{k=0}^{n}\lambda_k\bigg(\dfrac{h_2}{6}\mathscr{H}_x(g_y)_{i,0}^{n-k}-\dfrac{h_2^3}{90}\mathscr{H}_x\Big[(g_{yyy})_{i,0}^{n-k}-(g_{xxy})_{i,0}^{n-k}+(g_y)_{i,0}^{n-k}\\
\qquad+({}_0^CD_t^{\alpha+1}g_y)_{i,0}^{n-k}\Big]\bigg), \quad j=0\,, \quad 0\le i\le M_1\,,\\
0\,, \quad 1\le j\le M_2-1\,, \quad 0\le i\le M_1\,,\\
-\mu\sum\limits_{k=0}^{n}\lambda_k\bigg(\dfrac{h_2}{6}\mathscr{H}_x(g_y)_{i,M_2}^{n-k}-\dfrac{h_2^3}{90}\mathscr{H}_x\Big[(g_{yyy})_{i,M_2}^{n-k}-(g_{xxy})_{i,M_2}^{n-k}+(g_y)_{i,M_2}^{n-k}\\
\qquad+({}_0^CD_t^{\alpha+1}g_y)_{i,M_2}^{n-k}\Big]\bigg), \quad j=M_2\,, \quad 0\le i\le M_1\,.
\end{cases}
$$

Following the steps in the one-dimensional case, one can deduce

$$
\begin{aligned}
&\mathscr{H}(u_{ij}^{n+1}-u_{ij}^n)=\tau\mathscr{H}\phi_{ij}+\mu\Big[\sum_{k=0}^{n+1}\lambda_k(\Lambda-\mathscr{H})u_{ij}^{n+1-k}+\sum_{k=0}^{n}\lambda_k(\Lambda-\mathscr{H})u_{ij}^{n-k}\Big]\\
&\qquad\qquad+\frac{\tau}{2}\mathscr{H}(f_{ij}^n+f_{ij}^{n+1})+\frac{1}{2}(G_{ij}^n+G_{ij}^{n+1})+\tau(R_1)_{ij}^{n+1},\\
&u_{ij}^0=0\,, \qquad (x_i,y_j)\in\bar\Omega\,,
\end{aligned}
\tag{4.5}
$$

where $(R_1)_{ij}^{n+1}=O(\tau^2+h_1^4+h_2^4)$. Denoting $F_{ij}^n=\frac{1}{2}(\tau\mathscr{H}f_{ij}^n+G_{ij}^n)$, and adding a small term $\frac{\mu^2\lambda_0^2}{1+\mu\lambda_0}\delta_x^2\delta_y^2(u_{ij}^{n+1}-u_{ij}^n)=O(\tau^{3+2\alpha})$ on both sides of Eq. (4.5), we have

$$
\begin{aligned}
&\mathscr{H}(u_{ij}^{n+1}-u_{ij}^n)+\frac{\mu^2\lambda_0^2}{1+\mu\lambda_0}\delta_x^2\delta_y^2(u_{ij}^{n+1}-u_{ij}^n)\\
=&\tau\mathscr{H}\phi_{ij}+\mu\Big[\sum_{k=0}^{n+1}\lambda_k(\Lambda-\mathscr{H})u_{ij}^{n+1-k}+\sum_{k=0}^{n}\lambda_k(\Lambda-\mathscr{H})u_{ij}^{n-k}\Big]+F_{ij}^n+F_{ij}^{n+1}+\tau R_{ij}^{n+1}\\
&u_{ij}^0=0, \qquad (x_i,y_j)\in\bar\Omega\,,
\end{aligned}
\tag{4.6}
$$

with $R_{ij}^{n+1} = O(\tau^2 + h_1^4 + h_2^4)$. Omitting the truncation error in Eq. (4.6), we reach the following scheme in the ADI setting:

$$
\begin{aligned}
&\left(\sqrt{1+\mu\lambda_0}\mathscr{H}_x - \frac{\mu\lambda_0}{\sqrt{1+\mu\lambda_0}}\delta_x^2\right)\left(\sqrt{1+\mu\lambda_0}\mathscr{H}_y - \frac{\mu\lambda_0}{\sqrt{1+\mu\lambda_0}}\delta_y^2\right)u_{ij}^{n+1} \\
=&\mathscr{H}u_{ij}^n + \frac{\mu^2\lambda_0^2}{1+\mu\lambda_0}\delta_x^2\delta_y^2 u_{ij}^n + \mu\left[\sum_{k=1}^{n+1}\lambda_k(\Lambda-\mathscr{H})u_{ij}^{n+1-k} + \sum_{k=0}^{n}\lambda_k(\Lambda-\mathscr{H})u_{ij}^{n-k}\right] \\
&+\tau\mathscr{H}\phi_{ij} + F_{ij}^n + F_{ij}^{n+1}, \qquad (x_i,y_j)\in\Omega, \qquad 0\le n\le N-1, \\
&u_{ij}^0 = 0, \qquad (x_i,y_j)\in\bar{\Omega}.
\end{aligned}
$$

For ADI methods (cf. Ref. [27] for example), the solution $\{u_{ij}^{n+1}\}$ is determined by solving two independent one-dimensional problems. Specifically, the intermediate variables

$$
u_{ij}^* = \left(\sqrt{1+\mu\lambda_0}\mathscr{H}_y - \frac{\mu\lambda_0}{\sqrt{1+\mu\lambda_0}}\delta_y^2\right)u_{ij}^{n+1}, \qquad 0\le i\le M_1, \qquad 0\le j\le M_2,
$$

are first solved from the following system with fixed $j\in\{0,1,\cdots,M_2\}$:

$$
\begin{aligned}
&\left(\sqrt{1+\mu\lambda_0}\mathscr{H}_x - \frac{\mu\lambda_0}{\sqrt{1+\mu\lambda_0}}\delta_x^2\right)u_{ij}^* \\
=&\mathscr{H}u_{ij}^n + \frac{\mu^2\lambda_0^2}{1+\mu\lambda_0}\delta_x^2\delta_y^2 u_{ij}^n + \mu\left[\sum_{k=1}^{n+1}\lambda_k(\Lambda-\mathscr{H})u_{ij}^{n+1-k} + \sum_{k=0}^{n}\lambda_k(\Lambda-\mathscr{H})u_{ij}^{n-k}\right] \\
&+\tau\mathscr{H}\phi_{ij} + F_{ij}^n + F_{ij}^{n+1}, \qquad 0\le i\le M_1,
\end{aligned}
$$

When $\{u_{ij}^*\}$ is found, the approximate solution $\{u_{ij}^{n+1}\}$ is solved from the following system for fixed $i\in\{0,1,\cdots,M_1\}$:

$$
\left(\sqrt{1+\mu\lambda_0}\mathscr{H}_y - \frac{\mu\lambda_0}{\sqrt{1+\mu\lambda_0}}\delta_y^2\right)u_{ij}^{n+1} = u_{ij}^*, \qquad 0\le j\le M_2.
$$

Under the ADI method, the computational cost for solving a two-dimensional problem is usually greatly reduced.

We now proceed to give the convergence result of our compact ADI scheme (4.6), and one can show that the scheme is stable in the same sense as that given in Remark 3.1.

Theorem 4.1. *Assume that* $u(x,y,t)\in\mathscr{C}_{x,y,t}^{6,6,2}(\Omega\times[0,T])$ *is the solution of the problem* (4.1)–(4.3) *and* $\{u_{ij}^k|0\le i\le M_1,\ 0\le j\le M_2,\ 0\le k\le N\}$ *is a solution of the finite difference scheme* (4.6)*, respectively. Denote*

$$
e_{ij}^k = u(x_i,y_j,t_k) - u_{ij}^k, \quad 0\le i\le M_1, \quad 0\le j\le M_2, \quad 1\le k\le N.
$$

Then there exists a positive constant $\tilde{c}$ such that

$$\|e^k\| \leq \tilde{c}(\tau^2 + h_1^4 + h_2^4)\,, \quad 0 \leq k \leq N\,,$$

where $e^k = [e^k_{0,0}, e^k_{1,0}, \cdots, e^k_{M_1,0}, e^k_{0,1}, e^k_{1,1}, \cdots, e^k_{M_1,1}, \cdots, e^k_{0,M_2}, e^k_{1,M_2}, \cdots, e^k_{M_1,M_2}]^T$.

Proof. One can easily check that the following error equation holds:

$$\begin{aligned}
&(\bar{C}_{M_2+1} \otimes \bar{C}_{M_1+1})(e^{k+1} - e^k) + \frac{\mu^2\lambda_0^2}{(1+\mu\lambda_0)h_1^2h_2^2}(\bar{Q}_{M_2+1} \otimes \bar{Q}_{M_1+1})(e^{k+1} - e^k) \\
=& -\frac{\tau^{\alpha+1}}{2h_1^2}\sum_{l=0}^{k}\lambda_l(\bar{C}_{M_2+1} \otimes \bar{Q}_{M_1+1})(e^{k+1-l} + e^{k-l}) - \frac{\tau^{\alpha+1}}{2h_2^2}\sum_{l=0}^{k}\lambda_l(\bar{Q}_{M_2+1} \otimes \bar{C}_{M_1+1})(e^{k+1-l} + e^{k-l}) \\
&-\frac{\tau^{\alpha+1}}{2}\sum_{l=0}^{k}\lambda_l(\bar{C}_{M_2+1} \otimes \bar{C}_{M_1+1})(e^{k+1-l} + e^{k-l}) + \tau\bar{R}^{k+1}, \\
&e_{ij}^0 = 0\,, \qquad 0 \leq i \leq M_1\,, \qquad 0 \leq j \leq M_2\,,
\end{aligned} \tag{4.7}$$

where $\|\bar{R}^{k+1}\| \leq \tilde{c}_1(\tau^2 + h_1^4 + h_2^4)$ and the matrices $\bar{C}_{M_1+1},\ \bar{C}_{M_2+1},\ \bar{Q}_{M_1+1},\ \bar{Q}_{M_2+1}$ are given in (3.2), with the corresponding sizes given by the subscripts.

Multiplying the equation (4.7) with $(\frac{1}{2} \oplus I_{M_2-1} \oplus \frac{1}{2}) \otimes (\frac{1}{2} \oplus I_{M_1-1} \oplus \frac{1}{2})$, we get

$$\begin{aligned}
&(C_{M_2+1} \otimes C_{M_1+1})(e^{k+1} - e^k) + \frac{\mu^2\lambda_0^2}{(1+\mu\lambda_0)h_1^2h_2^2}(Q_{M_2+1} \otimes Q_{M_1+1})(e^{k+1} - e^k) \\
=& -\frac{\tau^{\alpha+1}}{2h_1^2}\sum_{l=0}^{k}\lambda_l(C_{M_2+1} \otimes Q_{M_1+1})(e^{k+1-l} + e^{k-l}) - \frac{\tau^{\alpha+1}}{2h_2^2}\sum_{l=0}^{k}\lambda_l(Q_{M_2+1} \otimes C_{M_1+1})(e^{k+1-l} + e^{k-l}) \\
&-\frac{\tau^{\alpha+1}}{2}\sum_{l=0}^{k}\lambda_l(C_{M_2+1} \otimes C_{M_1+1})(e^{k+1-l} + e^{k-l}) + \tau R^{k+1}, \\
&e_{ij}^0 = 0, \qquad 0 \leq i \leq M_1\,, \qquad 0 \leq j \leq M_2\,,
\end{aligned} \tag{4.8}$$

where $\|R^{k+1}\| \leq \tilde{c}_2(\tau^2 + h_1^4 + h_2^4)$ and

$$C_{M_1+1} = E^2_{M_1+1}, \quad C_{M_2+1} = E^2_{M_2+1}\,, \quad Q_{M_1+1} = S^T_{M_1+1}S_{M_1+1}\,, \quad Q_{M_2+1} = S^T_{M_2+1}S_{M_2+1}$$

are as in (3.4) and (3.5), respectively. (Once again, we have used the subscripts to indicate the sizes of the matrices.) We can now multiply (4.8) by $h_1h_2(e^{k+1} + e^k)^T$ and add up the equations for $0 \le k \le n-1$. Noting that

$$\begin{aligned}
&(e^{k+1}+e^k)^T(C_{M_2+1}\otimes C_{M_1+1})(e^{k+1}-e^k)\\
&\quad=(e^{k+1})^T(C_{M_2+1}\otimes C_{M_1+1})e^{k+1}-(e^k)^T(C_{M_2+1}\otimes C_{M_1+1})e^k,\\
&(e^{k+1}+e^k)^T(Q_{M_2+1}\otimes Q_{M_1+1})(e^{k+1}-e^k)\\
&\quad=(e^{k+1})^T(Q_{M_2+1}\otimes Q_{M_1+1})e^{k+1}-(e^k)^T(Q_{M_2+1}\otimes Q_{M_1+1})e^k\ ,\\
&h_1h_2(e^n)^T(C_{M_2+1}\otimes C_{M_1+1})e^n\ge\frac{1}{16}\|e^n\|^2,\\
&(e^n)^T(Q_{M_2+1}\otimes Q_{M_1+1})e^n\ge 0,\\
&(C_{M_2+1}\otimes Q_{M_1+1})=(E_{M_2+1}\otimes S^T_{M_1+1})(E_{M_2+1}\otimes S_{M_1+1})\ ,\\
&(Q_{M_2+1}\otimes C_{M_1+1})=(S^T_{M_2+1}\otimes E_{M_1+1})(S_{M_2+1}\otimes E_{M_1+1})\ ,
\end{aligned}$$

from Lemma 3.1 we have that

$$\begin{aligned}
&\frac{1}{16}\|e^n\|^2\\
\le&\tau h_1h_2(e^n+e^{n-1})^TR^n+\tau h_1h_2\sum_{k=0}^{n-2}(e^{k+1}+e^k)^TR^{k+1}\\
\le&\frac{1}{18}\|e^n\|^2+\frac{9\tau^2}{2}\|R^n\|^2+\frac{\tau}{2}\|e^{n-1}\|^2+\frac{\tau}{2}\|R^n\|^2+\frac{\tau}{2}\sum_{k=1}^{n-1}\|e^k\|^2+\frac{\tau}{2}\sum_{k=1}^{n-2}\|e^k\|^2+\tau\sum_{k=1}^{n-1}\|R^k\|^2\\
\le&\frac{1}{18}\|e^n\|^2+\frac{9\tau^2}{2}\|R^n\|^2+\tau\sum_{k=1}^{n-1}\|e^k\|^2+\tau\sum_{k=1}^{n}\|R^k\|^2\ ,
\end{aligned}$$

as in the one-dimensional case. □

5. Numerical Experiments

In this section, we describe numerical experiments for the finite difference scheme to illustrate our theoretical statements. All our tests were done in MATLAB. Although our theoretical results correspond to the discrete L^2 norm, we find that the maximum norm errors

$$E_\infty(h,\tau)=\max_{0\le k\le N}\|U^k-u^k\|_\infty$$

between the exact and the numerical solutions also match the proposed order for the examples we have tested (we had similar observations in Ref. [16]). In the numerical examples given below, the maximum norm errors are therefore reported.

We first consider the following one-dimensional problem.

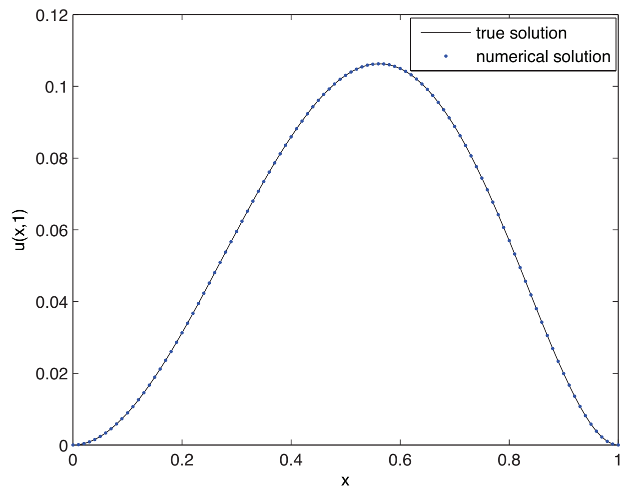

Figure 1: Exact solution and numerical solution for Example 5.1 at $t = 1$, when $\alpha = 0.5$ and $h = \tau = 1/100$.

Example 5.1.

$$
\begin{aligned}
&{}_0^C D_t^\gamma u = \frac{\partial^2 u}{\partial x^2} - u + g(x,t)\,, \quad 0 \le x \le 1\,, \quad 0 < t \le 1\,, \quad 1 < \gamma < 2\,,\\
&u(x,0) = 0\,, \quad \frac{\partial u(x,0)}{\partial t} = 0\,, \quad 0 \le x \le 1\,,\\
&\frac{\partial u(0,t)}{\partial x} = 0\,, \quad \frac{\partial u(L,t)}{\partial x} = 0\,, \quad 0 < t \le 1\,,
\end{aligned}
$$

where $g(x,t) = \frac{\Gamma(\gamma+3)}{2} t^2 e^x x^2 (1-x)^2 - e^x t^{\gamma+2}(2 - 8x + 8x^3)$. Note that the differential equation can be rewritten as

$$
\tfrac{\partial u(x,t)}{\partial t} = {}_0 I_t^\alpha [u_{xx}(x,t) - u(x,t)] + f(x,t)\,, \quad 0 \le x \le 1\,, \quad 0 < t \le 1\,,
$$

where $\alpha = \gamma - 1$, $f(x,t) = (\alpha+3)e^x x^2(1-x)^2 t^{\alpha+2} - \frac{\Gamma(\alpha+4)}{\Gamma(2\alpha+4)} e^x (2 - 8x + 8x^3) t^{2\alpha+3}$. The exact solution is $u(x,t) = e^x x^2 (1-x)^2 t^{\alpha+3}$.

Curves for the exact and numerical solutions for the problem at $t = 1$ with $\alpha = 0.5$ and $h = \tau = 1/100$ are shown in Fig. 1. The maximum norm errors are shown in Tables 1 and 2. The temporal convergence order and spatial convergence order, denoted by

$$
Rate1 = \log_2 \left(\frac{E_\infty(h, 2\tau)}{E_\infty(h,\tau)} \right) \quad \text{and} \quad Rate2 = \log_2 \left(\frac{E_\infty(2h, \tau)}{E_\infty(h,\tau)} \right),
$$

respectively, are also reported.

Let us now consider the stability of the scheme by testing (3.11) numerically. We note that the bound in (3.11) has been magnified to a certain extend when it is derived theoretically. In our test, we find that the quantity $B \doteq \left[\frac{5}{\Gamma(\alpha+1)} + 1\right] \sqrt{\|\delta_x^2 \rho\|^2 + \|\rho\|^2 + \|\tilde{\rho}\|^2}$ only

Table 1: Numerical convergence orders in temporal direction with $h=1/50$ for Example 5.1.

τ	$\alpha=0.3$		$\alpha=0.5$		$\alpha=0.7$	
	$E_\infty(h,\tau)$	Rate1	$E_\infty(h,\tau)$	Rate1	$E_\infty(h,\tau)$	Rate1
1/5	1.6417e-3	$*$	2.3844e-3	$*$	3.1904e-3	$*$
1/10	4.1558e-4	1.9820	6.0481e-4	1.9791	7.9961e-4	1.9964
1/20	1.0441e-4	1.9929	1.5221e-4	1.9904	2.0066e-4	1.9945
1/40	2.6115e-5	1.9993	3.8133e-5	1.9970	5.0254e-5	1.9975
1/80	6.4822e-6	2.0103	9.5080e-6	2.0038	1.2551e-5	2.0015

Table 2: Numerical convergence orders in spatial direction with $\tau=1/2000$ when $\alpha=0.5$ for Example 5.1.

h	$E_\infty(h,\tau)$	Rate2
1/2	2.2688e-2	$*$
1/4	1.3235e-3	4.0995
1/8	8.2429e-5	4.0050
1/16	5.1310e-6	4.0058
1/32	3.0926e-7	4.0524

Table 3: Stability of the scheme for Example 5.1 when $T=1$, $\alpha=0.5$.

$\rho=\tilde{\rho}=0.1x$				$\rho=\tilde{\rho}=0.1\sin(x)$			
$M=\frac{1}{h}$	$N=\frac{1}{\tau}$	$\|\varepsilon^N\|$	B	$M=\frac{1}{h}$	$N=\frac{1}{\tau}$	$\|\varepsilon^N\|$	B
100	500	0.3097	5.4637	100	500	0.2668	6.0461
	1000	0.3102	5.4637		1000	0.2672	6.0461
	2000	0.3105	5.4637		2000	0.2674	6.0461
	4000	0.3107	5.4637		4000	0.2675	6.0461
200	500	0.4349	7.6982	200	500	0.3747	8.5231
	1000	0.4356	7.6982		1000	0.3752	8.5231
	2000	0.4360	7.6982		2000	0.3755	8.5231
	4000	0.4363	7.6982		4000	0.3757	8.5231

can serve as a fine upper bound for $\|\varepsilon^N\|$. We have considered two kinds of perturbation given by the discretisation of some functions ρ and $\tilde{\rho}$, and the results are given in Table 3.

The next example is a two-dimensional problem.

Table 4: Numerical convergence orders in temporal direction with $h=\pi/50$ for Example 5.2.

τ	$\alpha=0.3$		$\alpha=0.5$		$\alpha=0.7$	
	$E_\infty(h,\tau)$	Rate1	$E_\infty(h,\tau)$	Rate1	$E_\infty(h,\tau)$	Rate1
1/5	1.5208e-2	*	2.0989e-2	*	2.9859e-2	*
1/10	3.7508e-3	2.0196	5.2056e-3	2.0115	7.5354e-3	1.9864
1/20	9.3437e-4	2.0051	1.2874e-3	2.0155	1.8780e-3	2.0045
1/40	2.3411e-4	1.9968	3.1958e-4	2.0102	4.6777e-4	2.0053
1/80	5.8774e-5	1.9939	7.9583e-5	2.0057	1.1666e-4	2.0035

Example 5.2.

$$
\begin{aligned}
&{}_0^C D_t^\gamma u = \Delta u - u + \cos(x)\cos(y)\left[\frac{\Gamma(\gamma+4)}{6}t^3 + 3t^{\gamma+3}\right],\\
&(x,y)\in\Omega=(0,\pi)\times(0,\pi),\quad 0<t\le 1,\\
&u(x,y,0)=0, \qquad \frac{\partial u(x,y,0)}{\partial t}=0, \qquad (x,y)\in\bar{\Omega},\\
&\frac{\partial u(x,y,t)}{\partial n}\Big|_{\partial\Omega}=0, \qquad (x,y)\in\partial\Omega, \qquad 0<t\le 1.
\end{aligned}
$$

Note that the differential equation can be rewritten as

$$
\frac{\partial u(x,y,t)}{\partial t} = {}_0I_t^\alpha(\Delta u - u) + \cos(x)\cos(y)\left[(\alpha+4)t^{\alpha+3} + \frac{3\Gamma(\alpha+5)}{\Gamma(2\alpha+5)}t^{2\alpha+4}\right],
$$

where $\alpha=\gamma-1$. The exact solution for this problem is $u(x,t)=\cos(x)\cos(y)t^{\alpha+4}$.

We let $h_1=h_2=h$ in this example. Fig. 2 shows the exact solution (left) and numerical solution (right) for Example 5.2 when $\alpha=0.5$ and $h=\tau=1/50$. In addition, the maximum norm errors between the exact and the numerical solutions

$$
E_\infty(h,\tau)=\max_{0\le k\le N}\max_{(x_i,y_j)\in\Omega}\left|u(x_i,y_j,t_k)-u_{ij}^k\right|
$$

are shown in Tables 4 and 5. The temporal convergence order and spatial convergence order, denoted by

$$
Rate1=\log_2\left(\frac{E_\infty(h,2\tau)}{E_\infty(h,\tau)}\right) \text{ and } Rate2=\log_2\left(\frac{E_\infty(2h,\tau)}{E_\infty(h,\tau)}\right),
$$

respectively, are again also reported. These tables confirm the theoretical analysis.

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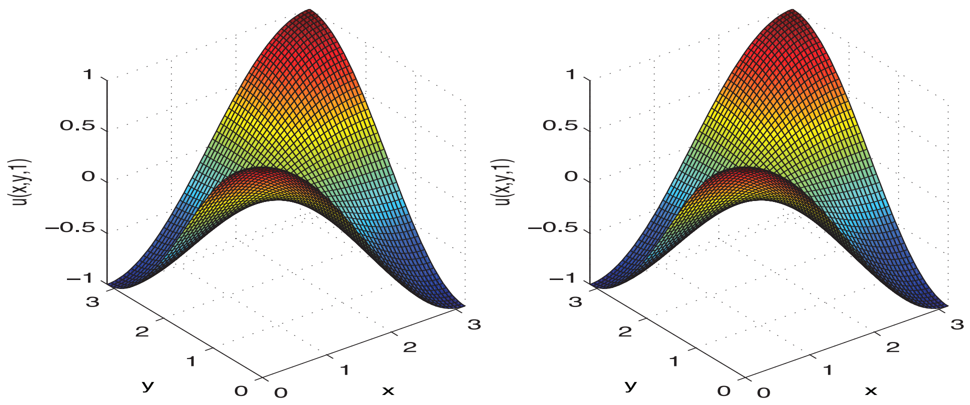

Figure 2: The exact solution (left) and numerical solution (right) for Example 5.2, when $\alpha = 0.5$, $h = \tau = 1/50$.

Table 5: Numerical convergence orders in spatial direction with $\tau = 1/20000$ when $\alpha = 0.5$ for Example 5.2.

h	$E_\infty(h,\tau)$	Rate2
$\pi/2$	3.4342e-3	$*$
$\pi/4$	2.0348e-4	4.0770
$\pi/8$	1.2502e-5	4.0247
$\pi/16$	7.7904e-7	4.0043
$\pi/32$	4.9832e-8	3.9665

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