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Ruin Probability in a Generalised Risk Process under Rates of Interest with Homogenous Markov Chains

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Abstract. This article explores recursive and integral equations for ruin probabilities of generalised risk processes, under rates of interest with homogenous Markov chain claims and homogenous Markov chain premiums. We assume that claim and premium take a countable number of non-negative values. Generalised Lundberg inequalities for the ruin probabilities of these processes are derived via a recursive technique. Recursive equations for finite time ruin probabilities and an integral equation for the ultimate ruin probability are presented, from which corresponding probability inequalities and upper bounds are obtained. An illustrative numerical example is discussed.

AMS subject classifications: 62P05, 60G40, 12E05

Key words: Integral equation, recursive equation, ruin probability, homogeneous Markov chain.

1. Introduction

Ruin probabilities in discrete time models have been considered by many authors. Teugels & Sundt [8, 9] studied the effects of a constant rate on the ruin probability under the compound Poisson risk model. Yang [11] established both exponential and non-exponential upper bounds for ruin probabilities in a risk model with constant interest force and independent premiums and claims. Xu & Wang [10] investigated a discrete-time risk model with constant interest force under a Markov chain interest rate. Yang & Zhang [12] considered a discrete-time insurance risk model by using an autoregressive process to model both the premuims and the claims, and they also included investment incomes in their model. Cai [1,2] investigated the ruin probabilities in two risk models, with independent premiums and claims and used a first-order autoregressive process to model the rates of interest. Cai & Dickson [3] obtained Lundberg inequalities for ruin probabilities in a two discrete-time risk process with a Markov chain interest model and independent premiums and claims. The author established Lundberg inequalities using a recursive technique for ruin probabilities in a two discrete-time risk process with homogenous Markov chain

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premiums when claims and rate of interest sequences are independent [5], and also by the Martingale approach in a two discrete-time risk process with homogenous Markov chain claims when premiums and rate of interest sequences are independent [6].

In this article, we extend the models considered by Cai & Dickson [3] to introduce homogenous Markov chain claims and homogenous Markov chain premiums, assuming independent rates of interest.

2. The Model and Basic Assumptions

Let $X = \{X_n\}_{n\geq 0}$ denote the premiums, $Y = \{Y_n\}_{n\geq 0}$ the claims and $I = \{I_n\}_{n\geq 0}$ the interests, where X, Y and I are defined on the probability space (Ω, A, P). To establish the probability inequalities for ruin probabilities, two styles of premium collection are considered. On the one hand, for premiums collected at the beginning of each period, the surplus process $\{U_n^{(1)}\}_{n\geq 1}$ with initial surplus u can be written

$$U_n^{(1)} = U_{n-1}^{(1)}(1 + I_n) + X_n - Y_n \,, \tag{2.1}$$

which can be rearranged as

$$U_n^{(1)} = u\prod_{k=1}^{n}(1 + I_k) + \sum_{k=1}^{n}\left(X_k - Y_k\right)\prod_{j=k+1}^{n}(1 + I_j)\,. \tag{2.2}$$

On the other hand, for premiums collected at the end of each period, the surplus process $\{U_n^{(2)}\}_{n\geq 1}$ with initial surplus u is

$$U_n^{(2)} = (U_{n-1}^{(2)} + X_n)(1 + I_n) - Y_n \,, \tag{2.3}$$

or equivalently

$$U_n^{(2)} = u\prod_{k=1}^{n}(1 + I_k) + \sum_{k=1}^{n}\left[X_k(1 + I_k) - Y_k\right]\prod_{j=k+1}^{n}(1 + I_j)\,, \tag{2.4}$$

where throughout this article $\prod_{t=a}^{b} z_t = 1$ and $\sum_{t=a}^{b} z_t = 0$ if $a > b$.

We make several assumptions.

Assumption 2.1. $U_o^{(1)} = U_o^{(2)} = u > 0$.

Assumption 2.2. $X = \{X_n\}_{n\geq 0}$ is an homogeneous Markov chain, such that for any n the values of X_n are taken from a set of non-negative numbers $E_X = \{x_1, x_2, \cdots, x_m, \cdots\}$ with $X_0 = x_i$ and

$$p_{ij} = P\left[\,\omega \in \Omega : X_{m+1}(\omega) = x_j \middle| X_m(\omega) = x_i\,\right], \quad (m \in N), \quad x_i, x_j \in E_X \,,$$

where $0 \leq p_{ij} \leq 1$, $\sum_{j=1}^{+\infty} p_{ij} = 1$.

Assumption 2.3. $Y = \{Y_n\}_{n\geq 0}$ is a homogeneous Markov chain, such that for any n the values of Y_n are taken from a set of non-negative numbers $E_Y = \{y_1, y_2, \cdots, y_n, \cdots\}$ with $Y_o = y_r$ and

$$q_{rs} = P\left[\omega \in \Omega : Y_{m+1}(\omega) = y_s \middle| Y_m(\omega) = y_r\right], \quad (m \in N), \quad y_r, y_s \in E_Y,$$

where $0 \leq q_{rs} \leq 1$, $\sum_{s=1}^{+\infty} q_{rs} = 1$.

Assumption 2.4. $I = \{I_n\}_{n\geq 0}$ is a sequence of independent and identically distributed non-negative continuous random variables with the same distributive function

$$F(t) = P(\omega \in \Omega : I_0(\omega) \leq t).$$

Assumption 2.5. X, Y and I are assumed to be independent.

For Eq. (2.1) with Assumptions 2.1 to 2.5, the finite time and ultimate ruin probabilities are defined by

$$\begin{aligned}&\psi_n^{(1)}(u, x_i, y_r)\\ =&P\left(\omega \in \Omega : \bigcup_{k=1}^{n} (U_k^{(1)}(\omega) < 0) \middle| U_o^{(1)}(\omega) = u, X_o(\omega) = x_i, Y_o(\omega) = y_r\right),\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\psi^{(1)}(u, x_i, y_r) = \lim_{n\to\infty} \psi_n^{(1)}(u, x_i, y_r)\\ =&P\left(\omega \in \Omega : \bigcup_{k=1}^{\infty} (U_k^{(1)}(\omega) < 0) \middle| U_o^{(1)}(\omega) = u, X_o(\omega) = x_i, Y_o(\omega) = y_r\right).\end{aligned} \tag{2.6}$$

On the other hand, for Eq. (2.3) with Assumptions 2.1 to 2.5, the finite time and ultimate ruin probabilities are defined by

$$\begin{aligned}&\psi_n^{(2)}(u, x_i, y_r)\\ =&P\left(\omega \in \Omega : \bigcup_{k=1}^{n} (U_k^{(2)}(\omega) < 0) \middle| U_o^{(2)}(\omega) = u, X_o(\omega) = x_i, Y_o(\omega) = y_r\right),\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\psi^{(2)}(u, x_i, y_r) = \lim_{n\to\infty} \psi_n^{(2)}(u, x_i, y_r)\\ =&P\left(\omega \in \Omega : \bigcup_{k=1}^{\infty} (U_k^{(2)}(\omega) < 0) \middle| U_o^{(2)}(\omega) = u, X_o(\omega) = x_i, Y_o(\omega) = y_r\right).\end{aligned} \tag{2.8}$$

We shall derive probability inequalities for $\psi^{(1)}(u, x_i, y_r)$ and $\psi^{(2)}(u, x_i, y_r)$. In Section 3, recursive equations are obtained for $\psi_n^{(1)}(u, x_i, y_r)$ and $\psi_n^{(2)}(u, x_i, y_r)$, and integral equations for $\psi^{(1)}(u, x_i, y_r)$ and $\psi^{(2)}(u, x_i, y_r)$. Probability inequalities for $\psi^{(1)}(u, x_i, y_r)$ and $\psi^{(2)}(u, x_i, y_r)$ are constructed in Section 4 by an inductive approach. An illustrative numerical example is then given in Section 5, and our conclusions in Section 6.

3. Integral Equation for Ruin Probabilities

We now construct a recursive equation for finite time ruin probabilities and an integral equation for the ultimate ruin probability, firstly a recursive equation for $\psi_n^{(1)}(u, x_i, y_r)$ and an integral equation for $\psi^{(1)}(u, x_i, y_r)$.

Theorem 3.1. *Given Eq.* (2.1) *and Assumptions* 2.1 *to* 2.5*, for* $n = 1, 2, \cdots$*, we have*

$$\begin{aligned}\psi_{n+1}^{(1)}(u, x_i, y_r) = \sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty} p_{ij} q_{rs} \Bigg\{ & F\left(\frac{y_s - x_j - u}{u}\right) \\ & + \int_{\frac{y_s - x_j - u}{u}}^{+\infty} \psi_n^{(1)}\left(u(1+t) + x_j - y_s, x_j, y_s\right) dF(t) \Bigg\},\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\psi^{(1)}(u, x_i, y_r) = \sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty} p_{ij} q_{rs} \Bigg\{ & F\left(\frac{y_s - x_j - u}{u}\right) \\ & + \int_{\frac{y_s - x_j - u}{u}}^{+\infty} \psi^{(1)}\left(u(1+t) + x_j - y_s, x_j, y_s\right) dF(t) \Bigg\}.\end{aligned} \tag{3.2}$$

Proof. Consider $X_1(\omega) = x_j \in E_X, Y_1(\omega) = y_s \in E_Y (\omega \in \Omega)$ and

$$\begin{aligned} B &= \left\{\omega \in \Omega : U_o^{(1)}(\omega) = u, X_o(\omega) = x_i, Y_o(\omega) = y_r\right\}, \\ A_{js} &= \left\{\omega \in \Omega : X_1(\omega) = x_j, Y_1(\omega) = y_s\right\}, \\ A_1 &= \left\{\omega \in \Omega : I_1(\omega) < \frac{Y_1(\omega) - X_1(\omega) - u}{u}\right\}, \\ A_2 &= \left\{\omega \in \Omega : I_1(\omega) \geq \frac{Y_1(\omega) - X_1(\omega) - u}{u}\right\}. \end{aligned}$$

From Eq. (2.1), $U_1^{(1)}(\omega) = u(1+I_1(\omega)) + x_j - y_s$ and $P(\omega \in \Omega : U_1^{(1)}(\omega) < 0 | A_1 \cap A_{js} \cap B) = 1$ such that

$$P\left(\omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k^{(1)}(\omega) < 0) \middle| A_1 \cap A_{js} \cap B\right) = 1. \tag{3.3}$$

In addition,

$$P\left(\omega \in \Omega : U_1^{(1)}(\omega) < 0 \middle| A_2 \cap A_{js} \cap B\right) = 0. \tag{3.4}$$

Let $\{\tilde{X}_n\}_{n\geq 0}$, $\{\tilde{Y}_n\}_{n\geq 0}$, $\{\tilde{I}_n\}_{n\geq 0}$ be independent copies of $\{X_n\}_{n\geq 0}$, $\{Y_n\}_{n\geq 0}$, $\{I_n\}_{n\geq 0}$ with $\tilde{X}_o(\omega) = X_1(\omega) = x_j$, $\tilde{Y}_o(\omega) = Y_1(\omega) = y_s$, $\tilde{I}_o(\omega) = I_1(\omega)$, $(\omega \in \Omega)$. Thus Eqs. (2.2) and

(3.4) imply

$$
\begin{aligned}
&P\left(\omega\in\Omega:\bigcup_{k=1}^{n+1}(U_k^{(1)}(\omega)<0)\middle|A_2\cap A_{js}\cap B\right)\\
=&P\left(\omega\in\Omega:\bigcup_{k=2}^{n+1}(U_k^{(1)}(\omega)<0)\middle|A_2\cap A_{js}\cap B\right)\\
=&P\left(\omega\in\Omega:\bigcup_{k=2}^{n+1}\left(\left[u(1+I_1(\omega))+x_j-y_s\right]\prod_{m=2}^{k}(1+I_m(\omega))+\sum_{m=2}^{k}\left(X_m(\omega)-Y_m(\omega)\right)\right.\right.\\
&\left.\left.\times\prod_{p=m+1}^{k}(1+I_p(\omega))<0\right)\middle|A_2\cap A_{js}\cap B\right)\\
=&P\left(\omega\in\Omega:\bigcup_{k=1}^{n}\left(\tilde{U}_o^{(1)}(\omega)\prod_{m=1}^{k}(1+\tilde{I}_m(\omega))+\sum_{m=1}^{k}\left(\tilde{X}_m(\omega)-\tilde{Y}_m(\omega)\right)\prod_{p=m+1}^{k}(1+\tilde{I}_p(\omega))\right.\right.\\
&\left.\left.<0\right)\middle|\left(\tilde{U}_o^{(1)}(\omega)=u(1+I_1(\omega))+x_j-y_s,\tilde{X}_o(\omega)=x_j,\tilde{Y}_o(\omega)=y_s\right)\cap A_2\cap B\right), \quad (3.5)
\end{aligned}
$$

Now Eq. (2.1) implies

$$
\psi_{n+1}^{(1)}(u,x_i,y_r)=P\left(\omega\in\Omega:\bigcup_{k=1}^{n+1}(U_k^{(1)}(\omega)<0)\middle|B\right),
$$

so we have

$$
\begin{aligned}
&\psi_{n+1}^{(1)}(u,x_i,y_r)\\
=&\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}P\left(\omega\in\Omega:\bigcup_{k=1}^{n+1}(U_k^{(1)}(\omega)<0)\middle|A_{js}\cap B\right)\\
=&\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}\left\{P\left\{\omega\in\Omega:\bigcup_{k=1}^{n+1}\left(U_k^{(1)}(\omega)<0\right)\middle|A_1\cap A_{js}\cap B\right\}P(A_1|B\cap A_{js})\right.\\
&\left.+P\left\{\omega\in\Omega:\bigcup_{k=1}^{n+1}\left(U_k^{(1)}(\omega)<0\right)\middle|A_2\cap A_{js}\cap B\right\}P(A_2|B\cap A_{js})\right\}. \quad (3.6)
\end{aligned}
$$

From Eq. (3.3)

$$
\begin{aligned}
&P\left\{\omega\in\Omega:\bigcup_{k=1}^{n+1}\left(U_k^{(1)}(\omega)<0\right)\middle|A_1\cap A_{js}\cap B\right\}.P(A_1|A_{js}\cap B)\\
=&P\left(\omega\in\Omega:I_1(\omega)<\frac{y_s-x_j-u}{u}\right)=\int_0^{\frac{y_s-x_j-u}{u}}dF(t),
\end{aligned}
$$

and from Eq. (3.5)

$$P\left\{\omega\in\Omega:\bigcup_{k=1}^{n+1}\left(U_k^{(1)}(\omega)<0\right)\Big|A_2\cap A_{js}\cap B\right\}.P(A_2|A_{js}\cap B)$$
$$=P\left\{\omega\in\Omega:\bigcup_{k=1}^{n}(\widetilde{U}_k^{(1)}(\omega)<0)\Big|\widetilde{U}_0^{(1)}(\omega)=u(1+I_1(\omega))+x_j-y_s,\widetilde{X}_o(\omega)=x_j,\widetilde{Y}_o(\omega)=y_s\right\}$$
$$\times P\left(\omega\in\Omega:I_1(\omega)\geq\frac{y_s-x_j-u}{u}\right)$$
$$=\int_{\frac{y_s-x_j-u}{u}}^{+\infty}\psi_n^{(1)}\left(u(1+t)+x_j-y_s,x_j,y_s\right)dF(t)\,,$$

therefore Eq. (3.6) may be written

$$\psi_{n+1}^{(1)}(u,x_i,y_r)=\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}\left\{\int_0^{\frac{y_s-x_j-u}{u}}dF(t)\right.$$
$$\left.+\int_{\frac{y_s-x_j-u}{u}}^{+\infty}\psi_n^{(1)}(u(1+t)+x_j-y_s,x_j,y_s)dF(t)\right\}. \tag{3.7}$$

When $n=0$, we have

$$\psi_1^{(1)}(u,x_i,y_r)=\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}F\left(\frac{y_s-x_j-u}{u}\right). \tag{3.8}$$

From the dominated convergence theorem, the integral equation for $\psi^{(1)}(u,x_i,y_r)$ in Theorem 3.1 then follows immediately by letting $n\to\infty$ in Eq. (3.7). □.

A recursive equation for $\psi_n^{(2)}(u,x_i,y_r)$ and an integral equation for $\psi^{(2)}(u,x_i,y_r)$ similarly hold, as stated in the following theorem.

Theorem 3.2. *Let model* (2.3) *satisfy Assumption* 2.1 *to Assumption* 2.5 *then for* $n=1,2,\cdots$

$$\psi_{n+1}^{(2)}(u,x_i,y_r)=\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}\left\{F\left(\frac{y_s-(x_j+u)}{u+x_j}\right)\right.$$
$$\left.+\int_{\frac{y_s-(x_j+u)}{u+x_j}}^{+\infty}\psi_n^{(2)}\left((u+x_j)(1+t)-y_s,x_j,y_s\right)dF(t)\right\}, \tag{3.9}$$

and

$$\psi^{(2)}(u,x_i,y_r)=\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}\left\{F\left(\frac{y_s-(x_j+u)}{u+x_j}\right)\right.$$
$$\left.+\int_{\frac{y_s-(x_j+u)}{u+x_j}}^{+\infty}\psi^{(2)}\left((u+x_j)(1+t)-y_s,x_j,y_s\right)dF(t)\right\}. \tag{3.10}$$

4. Probability Inequalities for Ruin Probabilities

We now establish probability inequalities for the ruin probabilities corresponding to Eq. (2.1) and Eq. (2.3), respectively. Thus for Eq. (2.1), we first prove the following Lemma.

Lemma 4.1. *Given* (2.1) *and Assumptions* 2.1 *to* 2.5*, and*

$$E(Y_1 \big| \omega \in \Omega : Y_o(\omega) = y_r) < E\left(X_1 \big| \omega \in \Omega : X_o(\omega) = x_i\right)$$

and

$$P\left((Y_1 - X_1) > 0 \big| \omega \in \Omega : X_o(\omega) = x_i, Y_o(\omega) = y_r\right) > 0 \tag{4.1}$$

for any $x_i \in E_X$ and $y_r \in E_Y$, then there exists a unique positive constant R_{ir} satisfying

$$E\left(e^{R_{ir}(Y_1 - X_1)} \Big| \omega \in \Omega : X_o(\omega) = x_i, Y_o(\omega) = y_r\right) = 1 \, . \tag{4.2}$$

Proof. Let $f_{ir}(t) = E\{e^{t(Y_1 - X_1)} | \omega \in \Omega : X_o(\omega) = x_i, Y_o(\omega) = y_r\} - 1$; $t \in (0, +\infty)$, when

$$\begin{aligned} f_{ir}(t) &= E\left\{e^{tY_1} \big| \omega \in \Omega : Y_o(\omega) = y_r\right\} . E\left\{e^{-tX_1} \big| \omega \in \Omega : X_o(\omega) = x_i\right\} - 1 \\ &= g_r(t).h_i(t) - 1 \, . \end{aligned}$$

As Y_1 is a discrete random variable taking values in $E_Y = \{y_1, y_2, \cdots, y_n, \cdots\}$, we have that

$$g_r(t) = E\left\{e^{tY_1} \big| \omega \in \Omega : Y_o(\omega) = y_r\right\} = \sum_{s=1}^{+\infty} q_{rs} e^{ty_s}$$

has an n-th derivative function on $(0, +\infty)$ for any $n \in N^* = N \backslash \{0\}$. Similarly, as X_1 is a discrete random variable taking values in $E_X = \{x_1, x_2, \cdots, x_m, \cdots\}$, we also have that

$$h_i(t) = E\left\{e^{-tX_1} \big| \omega \in \Omega : X_o(\omega) = x_i\right\} - 1 = \sum_{j=1}^{+\infty} p_{ij} e^{-tx_j}$$

has an n-th derivative function on $(0, +\infty)$ for any $n \in N^* = N \backslash \{0\}$. Consequently, $f_{ir}(t)$ has an n-th derivative function on $(0, +\infty)$ (any $n \in N^* = N \backslash \{0\}$) and

$$\begin{aligned} f'_{ir}(t) &= E\left\{(Y_1 - X_1) e^{t(Y_1 - X_1)} \Big| \omega \in \Omega : X_o(\omega) = x_i, Y_o(\omega) = y_r\right\} , \\ f''_{ir}(t) &= E\left\{(Y_1 - X_1)^2 e^{t(Y_1 - X_1)} \Big| \omega \in \Omega : X_o(\omega) = x_i, Y_o(\omega) = y_r\right\} \geq 0 \, , \end{aligned}$$

which implies that

$$f_{ir}(t) \text{ is a convex function with } f_{ir}(0) = 0 \tag{4.3}$$

and

$$\begin{aligned} f'_{ir}(0) &= E\left\{(Y_1 - X_1) \big| \omega \in \Omega : X_o(\omega) = x_i, Y_o(\omega) = y_r\right\} \\ &= E\left(Y_1 \big| \omega \in \Omega : Y_o(\omega) = y_r\right) - E(X_1 \big| \omega \in \Omega : X_o(\omega) = x_i) < 0 \, . \end{aligned} \tag{4.4}$$

As $P((Y_1 - X_1) > 0|\omega \in \Omega : X_o(\omega) = x_i, Y_o(\omega) = y_r) > 0$, we can find some constant $\delta > 0$ such that

$$P\left((Y_1 - X_1) > \delta > 0 \middle| \omega \in \Omega : X_o(\omega) = x_i, Y_o(\omega) = y_r\right) > 0 .$$

We therefore have

$$\begin{aligned}
&f_{ir}(t) \\
=&E\left\{ e^{t(Y_1-X_1)} \middle| \omega \in \Omega : X_o(\omega) = x_i, Y_o(\omega) = y_r \right\} - 1 \\
\geq&E\left(\left\{e^{t(Y_1-X_1)}|\omega \in \Omega : X_o(\omega) = x_i, Y_o(\omega) = y_r\right\}.1_{\{(Y_1-X_1)>\delta|\omega\in\Omega:X_o(\omega)=x_i,Y_o(\omega)=y_r\}}\right) - 1 \\
\geq&e^{t\delta}.P\left\{(Y_1 - X_1) > \delta \middle| \omega \in \Omega : X_o(\omega) = x_i, Y_o(\omega) = y_r\right\} - 1 ,
\end{aligned}$$

implying that

$$\lim_{t\to+\infty} f_{ir}(t) = +\infty , \tag{4.5}$$

and hence from Eqs. (4.3), (4.4) and (4.5) there exists a unique positive constant R_{ir} satisfying Eq. (4.2). □

Now consider $R_o = \inf\{R_{ir} > 0 : E(e^{R_{ir}(Y_1-X_1)}|\omega \in \Omega : X_o(\omega) = x_i, Y_o(\omega) = y_r) = 1(x_i \in E_X, y_r \in E_Y)\}$.

Remark 4.1. $E\left(e^{R_o(Y_1-X_1)} \middle| \omega \in \Omega : X_o(\omega) = x_i, Y_o(\omega) = y_r\right) \leq 1$.

Using Lemma 4.1 and Theorem 3.1, we obtain a probability inequality for $\psi^{(1)}(u, x_i, y_r)$ by an inductive approach as follows.

Theorem 4.1. *Given Eq.* (2.1) *and Assumptions* 2.1 *to* 2.5*, under the conditions of Lemma* 4.1 *and* $R_o > 0$ *we have that*

$$\psi^{(1)}(u, x_i, y_r) \leq \beta_1.E\left[^{-R_o u(1+I_1)}\right] \tag{4.6}$$

for any $u > 0,\ x_i \in E_X$ *and* $y_r \in E_Y$*, where*

$$\beta_1^{-1} = \inf_{\substack{z>0\\u>0}} \frac{e^{R_o uz}\int_0^z e^{-R_o ut}dF(t)}{F(z)}, \qquad 0 < \beta_1 \leq 1 .$$

Proof. Firstly, we have

$$\beta_1^{-1} = \inf_{\substack{z>0\\u>0}} \frac{\int_0^z e^{R_o u(z-t)}dF(t)}{F(z)} \geq \inf_{\substack{z>0\\u>0}} \frac{\int_0^z dF(t)}{F(z)} = 1 \Longleftrightarrow \beta_1 \leq 1 .$$

For any $z > 0$, we also have

$$\begin{aligned}
F(z) =& \left[\frac{e^{R_o uz}.\int_0^z e^{-R_o ut}dF(t)}{F(z)}\right]^{-1}.e^{R_o uz}.\int_0^z e^{-R_o ut}dF(t) \\
\leq&\beta_1.e^{R_o uz}.\int_0^z e^{-R_o ut}dF(t) .
\end{aligned} \tag{4.7}$$

From Eqs. (3.8) and (4.7), for any $u>0, x_i \in E_X$ and $y_r \in E_Y$ we then have $F((y_s - x_j - u)/u) = 0$ if $y_s \leq x_j + u$, whence

$$\psi_1^{(1)}(u, x_i, y_r) = 0 \leq \beta_1 E\left[e^{-R_o u(1+I_1)}\right] .$$

If $y_s > x_j + u$ then

$$\begin{aligned}
\psi_1^{(1)}(u, x_i, y_r) &= \sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty} p_{ij} q_{rs} F\left(\frac{y_s - x_j - u}{u}\right) \\
&\leq \beta_1 \sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty} p_{ij} q_{rs} e^{R_o\left[y_s - x_j - u\right]} . \int_0^{\frac{y_s - x_j - u}{u}} e^{-R_o u t} dF(t) \\
&= \beta_1 \sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty} p_{ij} q_{rs} e^{R_o(y_s - x_j)} . \int_0^{\frac{y_s - x_j - u}{u}} e^{-R_o u(1+t)} dF(t) \\
&\leq \beta_1 \sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty} p_{ij} q_{rs} e^{R_o(y_s - x_j)} . \int_0^{+\infty} e^{-R_o u(1+t)} dF(t) \\
&= \beta_1 E\left[e^{R_o(Y_1 - X_1)} \middle| \omega \in \Omega : X_o(\omega) = x_i, Y_o(\omega) = y_r \right] .E\left[e^{-R_o u(1+I_1)}\right] \\
&\leq \beta_1 E\left[e^{-R_o u(1+I_1)}\right] . \qquad (4.8)
\end{aligned}$$

Under an inductive hypothesis, we assume

$$\psi_n^{(1)}(u, x_i, y_r) \leq \beta_1 E\left[e^{-R_o u(1+I_1)}\right] , \qquad (4.9)$$

so inequality (4.8) implies (4.9) holds with $n = 1$. We have

$$\begin{aligned}
\psi_n^{(1)}\left(u(1+t) + x_j - y_s, x_j, y_s\right) &\leq \beta_1^* E\left[e^{-R_o^*\left[u(1+t)+x_j-y_s\right](1+I_1)}\right] \\
&\leq \beta_1^* e^{-R_o^*\left[u(1+t)+x_j-y_s\right]} . \qquad (4.10)
\end{aligned}$$

For $x_j \in E_X, y_s \in E_Y$, $u(1+t) + x_j - y_s > 0$ and $I_1(\omega) \geq 0, (\omega \in \Omega)$, where

$$\beta_1^{*-1} = \inf_{\substack{z>0 \\ u(1+t)+x_j-y_s>0}} \frac{e^{R_o^*[u(1+t)+x_j-y_s]z} \int_0^z e^{-R_o^*[u(1+t)+x_j-y_s]x} dF(x)}{F(z)},$$

$$R_o^* = \inf\left\{R_{js} > 0 : E\left(e^{R_{js}(Y_1 - X_1)} \middle| X_o = x_j, Y_o = y_s \right) = 1\right\}.$$

We have $R_o^* = R_o$ and $\beta_1^* = \beta_1$.
Thus as $R_o^*[u(1+t) + x_j - y_s] = R_o[u(1+t) + x_j - y_s] > 0$, inequality (4.10) may be rewritten

$$\psi_n^{(1)}\left(u(1+t) + x_j - y_s, x_j, y_s\right) \leq \beta_1 e^{-R_o\left[u(1+t)+x_j-y_s\right]} , \qquad (4.11)$$

so from Lemma 4.1, Eq. (3.1) and inequalities (4.7) and (4.11) we obtain

$$\begin{aligned}
&\psi^{(1)}_{n+1}(u,x_i,y_r)\\
=&\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}\left\{F\left(\frac{y_s-x_j-u}{u}\right)+\int_{\frac{y_s-x_j-u}{u}}^{+\infty}\psi^{(1)}_n\left(u(1+t)+x_j-y_s,x_j,y_s\right)dF(t)\right\}\\
\le&\beta_1\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}\left\{\int_0^{\frac{y_s-x_j-u}{u}}e^{R_o u\left[\frac{y_s-x_j-u}{u}-t\right]}dF(t)+\int_{\frac{y_s-x_j-u}{u}}^{+\infty}e^{-R_o\left[u(1+t)+x_j-y_s\right]}dF(t)\right\}\\
=&\beta_1\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}\left\{\int_0^{\frac{y_s-x_j-u}{u}}e^{R_o u\left[\frac{y_s-x_j-u(1+t)}{u}\right]}dF(t)+\int_{\frac{y_s-x_j-u}{u}}^{+\infty}e^{-R_o\left[u(1+t)+x_j-y_s\right]}dF(t)\right\}\\
=&\beta_1\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}\left\{\int_0^{\frac{y_s-x_j-u}{u}}e^{R_o\left[y_s-x_j-u(1+t)\right]}dF(t)+\int_{\frac{y_s-x_j-u}{u}}^{+\infty}e^{-R_o\left[u(1+t)+x_j-y_s\right]}dF(t)\right\}\\
=&\beta_1\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}e^{R_o(y_s-x_j)}\int_0^{+\infty}e^{-R_o u(1+t)}dF(t)\\
=&\beta_1 E\left[e^{R_o(Y_1-X_1)}\Big|\,\omega\in\Omega:X_o(\omega)=x_i,Y_o(\omega)=y_r\right].E\left[e^{-R_o u(1+I_1)}\right]\\
\le&\beta_1 E\left[e^{-R_o u(1+I_1)}\right].
\end{aligned}$$

Consequently

$$\psi^{(1)}_{n+1}(u,x_i,y_r)\le\beta_1 E\left[e^{-R_o u(1+I_1)}\right],$$

such that inequality (4.9) holds for any $n=1,2,\cdots$ and inequality (4.6) follows by letting $n\to\infty$ in inequality (4.9). □

Remark 4.2. Let $A(u,x_i,y_r)=\beta_1.E[e^{-R_o u(1+I_1)}]$. From $I_1(\omega)\ge 0$ $(\omega\in\Omega)$ and $\beta_1\le 1$, we have

$$A(u,x_i,y_r)\le\beta_1.E[e^{-R_o u}]=\beta_1 e^{-R_o u}\le e^{-R_o u},$$

so an upper bound for the ruin probability from inequality (4.6) is better than $e^{-R_o u}$.

Similar to Lemma 4.1, we have the following lemma.

Lemma 4.2. *Given* (2.3) *and Assumptions* 2.1 *to* 2.5*,* $E(I_1^k)<+\infty$ $(k=1,2)$*, and*

$$E\left[(Y_1-X_1(1+I_1)\Big|\,\omega\in\Omega:X_o(\omega)=x_i,Y_o(\omega)=y_r\right]<0$$

and

$$P\left(Y_1-X_1(1+I_1)>0\Big|\,\omega\in\Omega:X_o(\omega)=x_i,Y_o(\omega)=y_r\right)>0. \tag{4.12}$$

for any $x_i \in E_X$ and $y_r \in E_Y$, there exists a unique positive constant R_{ir} satisfying

$$E\left(e^{R_{ir}\left[Y_1 - X_1(1+I_1)\right]} \middle| \omega \in \Omega : X_o(\omega) = x_i, Y_o(\omega) = y_r \right) = 1. \tag{4.13}$$

Moreover, we obtain the following outcomes if we now let

$$\begin{aligned}\overline{R}_o =& \inf\left\{ R_{ir} > 0 : E\left(e^{R_{ir}\left[Y_1 - X_1(1+I_1)\right]} \middle| \omega \in \Omega : X_o(\omega) = x_i, Y_o(\omega) = y_r \right) \right.\\ &\left. = 1 (x_i \in E_X, y_r \in E_Y) \right\}.\end{aligned}$$

Remark 4.3. $E(e^{\overline{R}_o[Y_1 - X_1(1+I_1)]} | \, \omega \in \Omega : X_o(\omega) = x_i, Y_o(\omega) = y_r) \leq 1.$

Lemma 4.2 and Theorem 3.2 yields a probability inequality for $\psi^{(2)}(u, x_i, y_r)$ by an inductive approach.

Theorem 4.2. *Given Eq.* (2.3) *and Assumptions* 2.1 *to* 2.5*, under the conditions of Lemma* 4.2 *and $\overline{R}_o > 0$ we have*

$$\begin{aligned}&\psi^{(2)}(u, x_i, y_r)\\ \leq& \beta_2 E\left[e^{\overline{R}_o Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_r \right] E\left[e^{-\overline{R}_o (u + X_1)(1+I_1)} \middle| \omega \in \Omega : X_o(\omega) = x_i \right],\end{aligned} \tag{4.14}$$

for any $x_i \in E_X$ and $y_r \in E_Y$, where

$$\beta_2^{-1} = \inf_{\substack{z>0\\u>0}} \frac{e^{\overline{R}_o uz} \int_0^z e^{-\overline{R}_o ut} dF(t)}{F(z)}, \qquad 0 < \beta_2 \leq 1\,. \tag{4.15}$$

Proof. As for Theorem 4.1, with $\beta_2 \leq 1$ and any $z > 0$ we have

$$F(z) \leq \beta_2 . e^{\overline{R}_o uz} . \int_0^z e^{-\overline{R}_o ut} dF(t) \tag{4.16}$$

— so for any $u > 0,\ x_i \in E_X$ and $y_r \in E_Y$, if $y_s \leq u + x_j$ then $F(\frac{y_s - (u+x_j)}{u+x_j}) = 0$, whence

$$\begin{aligned}&\psi_1^{(2)}(u, x_i, y_r) = 0\\ \leq& \beta_2 E\left[e^{\overline{R}_o Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_r \right] .E\left[e^{\overline{R}_o (u + X_1)(1+I_1)} \middle| \omega \in \Omega : X_o(\omega) = x_i \right].\end{aligned}$$

If $y_s > u + x_j$, then

$$\begin{aligned}
&\psi_1^{(2)}(u,x_i,y_r)\\
=&\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}F\left(\frac{y_s-(u+x_j)}{u+x_j}\right)\\
\leq&\beta_2\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}\int_0^{\frac{y_s-(u+x_j)}{u+x_j}}e^{\overline{R}_o u\left[\frac{y_s-(u+x_j)}{u+x_j}-t\right]}dF(t)\\
=&\beta_2\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}\int_0^{\frac{y_s-(u+x_j)}{u+x_j}}e^{\overline{R}_o u\left[\frac{y_s-(u+x_j)(1+t)}{u+x_j}\right]}dF(t)\\
\leq&\beta_2\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}\int_0^{\frac{y_s-(u+x_j)}{u+x_j}}e^{\overline{R}_o\left[y_s-(u+x_j)(1+t)\right]}dF(t)\\
\leq&\beta_2\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}\int_0^{+\infty}e^{\overline{R}_o\left[y_s-(u+x_j)(1+t)\right]}dF(t)\\
=&\beta_2E\left[e^{\overline{R}_oY_1}\middle|\,\omega\in\Omega:Y_o(\omega)=y_r\right].E\left[e^{-\overline{R}_o(u+X_1)(1+I_1)}\middle|\,\omega\in\Omega:X_o(\omega)=x_i\right],
\end{aligned}$$

hence

$$\begin{aligned}
&\psi_1^{(2)}(u,x_i,y_r)\\
\leq&\beta_2E\left[e^{\overline{R}_oY_1}\middle|\,\omega\in\Omega:Y_o(\omega)=y_r\right].E\left[e^{-\overline{R}_o(u+X_1)(1+I_1)}\middle|\,\omega\in\Omega:X_o(\omega)=x_i\right].
\end{aligned}\tag{4.17}$$

Under an inductive hypothesis, we assume that

$$\begin{aligned}
&\psi_n^{(2)}(u,x_i,y_r)\\
\leq&\beta_2E\left[e^{\overline{R}_oY_1}\middle|\,\omega\in\Omega:Y_o(\omega)=y_r\right].E\left[e^{-\overline{R}_o(u+X_1)(1+I_1)}\middle|\,\omega\in\Omega:X_o(\omega)=x_i\right].
\end{aligned}\tag{4.18}$$

Inequality (4.17) implies that inequality (4.18) holds for $n=1$.
For $x_j\in E_X, y_s\in E_Y$, $t>\frac{y_s-(u+x_j)}{u+x_j}$ and $I_1(\omega)\geq 0$ $(\omega\in\Omega)$, we have

$$\begin{aligned}
&\psi_n^{(2)}\left((u+x_j)(1+t)-y_s,x_j,y_s\right)\\
\leq&\beta_2^*E\left[e^{\overline{R}_o^*Y_1}\middle|\,\omega\in\Omega:Y_o(\omega)=y_s\right].E\left[e^{-\overline{R}_o^*\left[(u+x_j)(1+t)-y_s+X_1\right](1+I_1)}\middle|\,\omega\in\Omega:X_o(\omega)=x_j\right]\\
=&\beta_2^*E\left[e^{\overline{R}_o^*Y_1}\middle|\,\omega\in\Omega:Y_o(\omega)=y_s\right].E\left[e^{-\overline{R}_o^*\left[(u+x_j)(1+t)-y_s\right](1+I_1)-\overline{R}_o^*X_1(1+I_1)}\middle|\,\omega\in\Omega:X_o(\omega)=x_j\right]\\
\leq&\beta_2^*E\left[e^{\overline{R}_o^*Y_1}\middle|\,\omega\in\Omega:Y_o(\omega)=y_s\right].E\left[e^{-\overline{R}_o^*\left[(u+x_j)(1+t)-y_s\right]-\overline{R}_o^*X_1(1+I_1)}\middle|\,\omega\in\Omega:X_o(\omega)=x_j\right]\\
=&\beta_2^*E\left[e^{\overline{R}_o^*Y_1}\middle|\,\omega\in\Omega:Y_o(\omega)=y_s\right].E\left[e^{-\overline{R}_o^*X_1(1+I_1)}\middle|\,\omega\in\Omega:X_o(\omega)=x_j\right].e^{-\overline{R}_o^*\left[(u+x_j)(1+t)-y_s\right]}\\
=&\beta_2^*.e^{-\overline{R}_o^*\left[(u+x_j)(1+t)-y_s\right]},
\end{aligned}\tag{4.19}$$

where

$$\beta_2^{*-1} = \inf_{\substack{z>0 \\ (u+x_j)(1+t)-y_s>0}} \frac{e^{\overline{R}_o^*[(u+x_j)(1+t)-y_s]z} \int_0^z e^{-\overline{R}_o^*[(u+x_j)(1+t)-y_s]x} dF(x)}{F(z)} ,$$

$$\overline{R}_o^* = \inf\left\{ R_{js} : E\left(e^{R_{js}(Y_1 - X_1(1+I_1))} \middle| X_o = x_j, Y_o = y_s \right) = 1 \right\} .$$

We have $\overline{R}_o^* = \overline{R}_o$ and $\beta_2^* = \beta_2$.
Thus

$$\overline{R}_o^* \left[(u+x_j)(1+t) - y_s\right] = \overline{R}_o \left[(u+x_j)(1+t) - y_s\right] > 0$$

such that

$$\psi_n^{(2)} \left((u+x_j)(1+t) - y_s, x_j, y_s\right) \le \beta_2 . e^{-\overline{R}_o\left[(u+x_j)(1+t)-y_s\right]} , \tag{4.20}$$

whence from Lemma 4.2, Eq. (3.9) and inequalities (4.9), (4.16) and (4.20) we obtain

$$\begin{aligned}
\psi_{n+1}^{(2)}(u, x_i, y_r) = & \sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty} p_{ij} q_{rs} \left\{ F\left(\frac{y_s - (u+x_j)}{u+x_j} \right) \right. \\
& \left. + \int_{\frac{y_s-(u+x_j)}{u+x_j}}^{+\infty} \psi_n^{(2)} \left((u+x_j)(1+t) - y_s, x_j, y_s\right) dF(t) \right\} \\
\le & \sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty} p_{ij} q_{rs} \left\{ \beta_2 \int_0^{\frac{y_s-(u+x_j)}{u+x_j}} e^{\overline{R}_o u\left[\frac{y_s-(u+x_j)}{u+x_j} - t\right]} dF(t) \right. \\
& \left. + \beta_2 \int_{\frac{y_s-(u+x_j)}{u+x_j}}^{+\infty} e^{-\overline{R}_o\left[(u+x_j)(1+t)-y_s\right]} dF(t) \right\} \\
= & \sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty} p_{ij} q_{rs} \left\{ \beta_2 \int_0^{\frac{y_s-(u+x_j)}{u+x_j}} e^{\overline{R}_o u\left[\frac{y_s-(u+x_j)(1+t)}{u+x_j}\right]} dF(t) \right. \\
& \left. + \beta_2 \int_{\frac{y_s-(u+x_j)}{u+x_j}}^{+\infty} e^{-\overline{R}_o\left[(u+x_j)(1+t)-y_s\right]} dF(t) \right\} \\
\le & \sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty} p_{ij} q_{rs} \left\{ \beta_2 \int_0^{\frac{y_s-(u+x_j)}{u+x_j}} e^{-\overline{R}_o\left[(u+x_j)(1+t)-y_s\right]} dF(t) \right. \\
& \left. + \beta_2 \int_{\frac{y_s-(u+x_j)}{u+x_j}}^{+\infty} e^{-\overline{R}_o\left[(u+x_j)(1+t)-y_s\right]} dF(t) \right\}
\end{aligned}$$

$$=\beta_2 \sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty} p_{ij}q_{rs} \int_0^{+\infty} e^{\overline{R}_o\left[y_s-(u+x_j)(1+t)\right]} dF(t)$$
$$=\beta_2 E\left[e^{\overline{R}_o Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_r \right] .E\left[e^{-\overline{R}_o(u+X_1)(1+I_1)} \middle| \omega \in \Omega : X_o(\omega) = x_i \right],$$

whence

$$\psi^{(2)}_{n+1}(u, x_i, y_r)$$
$$\leq \beta_2 E\left[e^{\overline{R}_o Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_r \right] .E\left[e^{-\overline{R}_o(u+X_1)(1+I_1)} \middle| \omega \in \Omega : X_o(\omega) = x_i \right] .$$

Thus we have inequality (4.18) for any $n = 1, 2, \cdots$, and inequality (4.14) follows by letting $n \to \infty$. □

Remark 4.4. Let

$$B(u, x_i, y_r) = \beta_2 E\left[e^{\overline{R}_o Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_r \right] .E\left[e^{-\overline{R}_o(u+X_1)(1+I_1)} \middle| \omega \in \Omega : X_o(\omega) = x_i \right].$$

From $I_1(\omega) \geq 0$, $X_1(\omega) \geq 0$ $(\omega \in \Omega)$ and $\beta_2 \leq 1$, we have

$$B(u, x_i, y_r)$$
$$=\beta_2 E\left[e^{\overline{R}_o Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_r \right] E\left[e^{-\overline{R}_o u(1+I_1)-\overline{R}_o X_1(1+I_1)} \middle| \omega \in \Omega : X_o(\omega) = x_i \right]$$
$$\leq \beta_2 E\left[e^{\overline{R}_o Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_r \right] E\left[e^{-\overline{R}_o u-\overline{R}_o X_1(1+I_1)} \middle| \omega \in \Omega : X_o(\omega) = x_i \right]$$
$$=\beta_2 E\left[e^{\overline{R}_o Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_r \right] E\left[e^{-\overline{R}_o X_1(1+I_1)} \middle| \omega \in \Omega : X_o(\omega) = x_i \right] .e^{-\overline{R}_o u}$$
$$\leq \beta_2 e^{-\overline{R}_o u} \leq e^{-\overline{R}_o u},$$

so the upper bound for the ruin probability in inequality (4.14) is better than $e^{-\overline{R}_o u}$.

5. Numerical Example

We now give a numerical example to illustrate the bounds of $\psi^{(1)}(u, x_i, y_r)$ derived in Section 4. Let $X = \{X_n\}_{n\geq 0}$ be an homogeneous Markov chain such that X_n takes values in $E_X = \{2, 4\}$ for any n, with X_1 having distribution

X_1	2	4
P	0.65	0.35

and $P = [p_{ij}]_{2x2}$ given by

$$P = \begin{bmatrix} 0.4 & 0.6 \\ 0.35 & 0.65 \end{bmatrix} .$$

Let $Y = \{Y_n\}_{n\geq 0}$ be an homogeneous Markov chain such that Y_n takes values in $E_Y = \{2,4\}$ for any n, with Y_1 having distribution

Y_1	1	3
P	0.7	0.3

and $Q = [q_{ij}]_{2x2}$ given by

$$P = \begin{bmatrix} 0.45 & 0.55 \\ 0.5 & 0.5 \end{bmatrix} .$$

Then we have

$$E(X_1 \big| X_o = 2) = 3.2\,, \qquad E(X_1 \big| X_o = 4) = 3.3\,,$$
$$E(Y_1 \big| Y_o = 1) = 2.1\,, \qquad E(Y_1 \big| Y_o = 3) = 2.0\,.$$

such that

$$E\left(X_1 \big| X_o = x_i\right) > E\left(Y_1 \big| Y_o = y_r\right) \quad \forall x_i \in E_X, y_r \in E_Y\,, \tag{5.1}$$

and $Y_1 - X_1$ has distribution

$Y_1 - X_1$	-3	-1	1
P	0.245	0.56	0.195

.

Suppose $A_1 = \{X_o = 2; Y_o = 1\}$, $A_2 = \{X_o = 2; Y_o = 3\}$, $A_3 = \{X_o = 4; Y_o = 1\}$ and $A_4 = \{X_o = 4; Y_o = 3\}$. Then we have

$$P\left(Y_1 - X_1 > 0 \big| A_1\right) = 0.22 > 0\,, \qquad P\left(Y_1 - X_1 > 0 \big| A_2\right) = 0.18 > 0\,, \tag{5.2}$$
$$P\left(Y_1 - X_1 > 0 \big| A_3\right) = 0.1925 > 0\,, \qquad P\left(Y_1 - X_1 > 0 \big| A_4\right) = 0.175 > 0\,. \tag{5.3}$$

Inequalities (5.1), (5.2) and (5.3) imply Lemma 4.1 holds. Now $(Y_1 - X_1)|A_1$ has distribution

$Y_1 - X_1 \vert A_1$	-3	-1	1
P	0.27	0.51	0.22

,

and from Lemma 4.1 $R_1 > 0$ satisfies the equation

$$0.27e^{-3R_1} + 0.51e^{-R_1} + 0.22e^{R_1} = 1$$
$$\Leftrightarrow 22t^4 - 100t^3 + 51t^2 + 27 = 0 (t = e^{R_1})\,. \tag{5.4}$$

On solving Eq. (5.4) using Maple, we have

$$R_1 = \ln\left(\frac{1}{22}\sqrt[3]{31832 + 7590\sqrt{6}} + \frac{437}{11\sqrt[3]{31832 + 7590\sqrt{6}}} + \frac{13}{11}\right) \approx 1.37028\,.$$

Now $(Y_1 - X_1)|A_2$ has distribution

$Y_1 - X_1 \| A_2$	-3	-1	1
P	0.3	0.5	0.2

,

and from Lemma 4.1 $R_2 > 0$ satisfies the equation

$$0.3e^{-3R_2} + 0.5e^{-R_2} + 0.2e^{R_2} = 1$$
$$\Leftrightarrow 2t^4 - 10t^3 + 5t^2 + 3 = 0\,(t = e^{R_2}) \tag{5.5}$$

On solving Eq. (5.5) using Maple, we have

$$R_2 = \ln\left(\frac{1}{6}\sqrt[3]{890 + 18\sqrt{743}} + \frac{437}{3\sqrt[3]{890 + 18\sqrt{743}}} + \frac{4}{3}\right) \approx 4.41653\,.$$

Now $(Y_1 - X_1)|A_3$ has distribution

$Y_1 - X_1 \| A_3$	-3	-1	1
P	0.2925	0.515	0.1925

,

and from Lemma 4.1 $R_3 > 0$ satisfies the equation

$$0.2925e^{-3R_3} + 0.515e^{-R_3} + 0.1925e^{R_3} = 1$$
$$\Longleftrightarrow 1925t^4 - 10000t^3 + 5150t^2 + 2925 = 0\,(t = e^{R_3})\,. \tag{5.6}$$

On solving Eq. (5.6) using Maple, we have

$$R_3 = \ln\left(\frac{2}{231}\sqrt[3]{7019713 + 3465\sqrt{1154634}} + \frac{65678}{231\sqrt[3]{7019713 + 3465\sqrt{1154634}}} + \frac{323}{231}\right)$$
$$\approx 4.59722\,.$$

Finally, $(Y_1 - X_1)|A_4$ has distribution

$Y_1 - X_1 \| A_4$	-3	-1	1
P	0.325	0.5	0.175

,

and from Lemma 4.1 $R_4 > 0$ satisfies the equation

$$0.325e^{-3R_4} + 0,5e^{-R_4} + 0.175e^{R_4} = 1$$
$$\Leftrightarrow 175t^4 - 1000t^3 + 500t^2 + 325 = 0\,(t = e^{R_4})\,. \tag{5.7}$$

On solving Eq. (5.7) using Maple, we have

$$R_4 = \ln\left(\frac{1}{21}\sqrt[3]{58050 + 42\sqrt{478023}} + \frac{454}{7\sqrt[3]{58050 + 42\sqrt{478023}}} + \frac{11}{7}\right) \approx 5.14537.$$

Table 1: Upper bounds $C(u,\lambda)$ of $\psi^{(1)}(u,x_i,y_r)$

u	$\lambda = 1$	$\lambda = 0.5$	$\lambda = 0.25$
1	0.107175447	0.135827694	0.429254335
1.5	0.041905547	0.050104894	0.410015675
2	0.01725255	0.01991452	0.399954849
2.5	0.00734946	0.00828553	0.393771005
3	0.003207691	0.003555534	0.389585629
3.5	0.001425626	0.001560221	0.38656486
4	0.000642584	0.000696301	0.384282043
4.5	0.00029291	0.00031488	0.382496226
5	0.00013475	0.000143915	0.381061052
5.5	$6.24655.10^{-5}$	$6.63519.10^{-5}$	0.379882489
6	$2.91447.10^{-5}$	$3.08155.10^{-5}$	0.378897365

Consequently, $R_o = \min\{R_1, R_2, R_3, R_4\} = R_1 \approx 1,37028$.

Let $I = \{I_n\}_{n\geq 0}$ be a sequence of independent and identically distributed (i.i.d) non-negative random variable with distribution function $F(t) = 1 - e^{-\lambda t}(t \geq 0)$. We can apply the result of Theorem 4.1 for $\psi^{(1)}(u,x_i,y_r)$ to obtain

$$\psi^{(1)}(u,x_i,y_r) \leq \beta_1 E\left[e^{-R_o u(1+I_1)}\right] \leq E\left[e^{-R_o u(1+I_1)}\right] = C(u,\lambda)\,, \tag{5.8}$$

where

$$C(u,\lambda) = \int_0^{+\infty} e^{-R_o u-(R_o u+\lambda)t}dt = \frac{e^{-R_o u}}{R_o u + \lambda}\,. \tag{5.9}$$

Table 1 shows upper bound values $C(u,\lambda)$ of $\psi^{(1)}(u,x_i,y_r)$, for a range of values of u and λ.

6. Conclusion

Theorems 4.1 and 4.2 provide upper bounds for $\psi^{(1)}(u,x_i,y_r)$ and $\psi^{(2)}(u,x_i,y_r)$, by using a recursive technique. To reach these theorems, we began by obtaining important preliminary results — viz. Theorems 3.1 and 3.2, which give recursive equations for finite time ruin probabilities and integral equations for ultimate ruin probability. In addition, we obtained Lemmas 4.1 and 4.2, which give Lundbergs constants. Our results were illustrated in an application to the ruin probability for a risk process with $X = \{X_n\}_{n\geq 0}$ and $Y = \{Y_n\}_{n\geq 0}$ homogeneous Markov chains, and $I = \{I_n\}_{n\geq 0}$ a sequence of independent and identically distributed (i.i.d) non-negative random variable distribution functions

$F(t) = 1 - e^{-\lambda t}(t \geq 0)$.

There remain many open issues — e.g.

(a) building upper bounds for $\psi^{(1)}(u, x_i, y_r)$ and $\psi^{(2)}(u, x_i, y_r)$ by the martingale approach;
(b) extending results of this article to consider $X = \{X_n\}_{n\geq 0}$ and $Y = \{Y_n\}_{n\geq 0}$ homogeneous Markov chains, and $I = \{I_n\}_{n\geq 0}$ a first- order autoregressive process; and
(c) letting $\tau_m := \inf\{k \geq 1 | U_k^{(m)} < 0\}$ be the time of ruin, and calculating or estimating quantities such as $E(\tau_m)$.

Further research in some of these directions is in progress.

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