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Semilocal Convergence Theorem for a Newton-like Method

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Abstract. The semilocal convergence of a third-order Newton-like method for solving nonlinear equations is considered. Under a weak condition (the so-called γ-condition) on the derivative of the nonlinear operator, we establish a new semilocal convergence theorem for the Newton-like method and also provide an error estimate. Some numerical examples show the applicability and efficiency of our result, in comparison to other semilocal convergence theorems.

AMS subject classifications: 65F10, 65F50, 65H10

Key words: Newton-like method, nonlinear equation, Newton-Kantorovich theorem, γ-condition, error estimate.

1. Introduction

Finding solutions of the system

$$F(x)=0 \tag{1.1}$$

is widely studied theoretically and in many applications, where F is a nonlinear operator often defined on an open convex Ω of a Banach space X with values in a Banach space Y. Iterative numerical methods are often used and the most famous is the Newton method where

$$x_{n+1}=x_n-F'(x_n)^{-1}F(x_n), \quad n\geqslant 0, \quad x_0\in\Omega, \tag{1.2}$$

which converges quadratically under reasonable conditions.

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Many researchers have focused on the convergence properties of the Newton method. One famous result is the well-known Kantorovich theorem [4], which guarantees the convergence of the sequence $\{x_n\}$ to a solution $\{x^*\}$ under mild conditions. Subsequently, many Kantorovich-type convergence theorems were obtained — cf. Refs. [1–3, 5, 6]. Thus in considering the computing complexity, Smale presented the α-theory at the 20th International Conference of Mathematicians (cf. Ref. [5]); and Wang [6] proposed a weak condition (the so-called γ—condition) on the derivative of the nonlinear operator, which improved Smale's theorem.

On the other hand, many Newton-like methods have also been proposed to improve the convergence [7–11, 18–28]. For example, Frontini presented a Newton-like method where

$$\begin{cases} y_n = x_n - f'(x_n)^{-1} f(x_n), \\ x_{n+1} = x_n - f'\left(\frac{x_n+y_n}{2}\right)^{-1} f(x_n), \end{cases} \tag{1.3}$$

where f is a real or complex function; and Weerakoon & Fernando presented a Newton-like method with third order where

$$\begin{cases} y_n = x_n - f'(x_n)^{-1} f(x_n), \\ x_{n+1} = x_n - 2\big[f'(x_n) + f'(y_n)\big]^{-1} f(x_n). \end{cases} \tag{1.4}$$

They and others [12–17] extended these methods to Banach space, and also established relevant Newton-Kantorovich theorems under the γ-condition.

Homeier [18] presented an iterative method

$$x_{n+1} = x_n - f(x_n) \sum_{r=1}^{m} \omega_r \frac{1}{f'(x_n - \tau_r f(x_n)/f'(x_n))},$$

where f is a real smooth function with a simple zero and ω_r, τ_r satisfy $\sum_{r=1}^{m} \omega_r = 1$, $\sum_{r=1}^{m} \omega_r \tau_r = 1/2$. He proved that the sequence $\{x_n\}$ generated by this method converges to the solution x^* cubically. For $m = 2$, $\omega_1 = \omega_2 = 1/2$ and $\tau_1 = 1 - \tau_2 = 0$ we have

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(x_n - f(x_n)/f'(x_n))}\right). \tag{1.5}$$

We generalise this Newton-like method for solving Eq. (1.1) in Banach space. If $F(x)$ is Fréchet differentiable in Ω and $F'(x)^{-1}$ exists, the iteration (1.5) can be reformulated as

$$\begin{cases} y_n = x_n - F'(x_n)^{-1} F(x_n), \\ x_{n+1} = x_n - \frac{1}{2}\big[F'(x_n)^{-1} + F'(y_n)^{-1}\big] F(x_n). \end{cases} \tag{1.6}$$

We establish a semilocal convergence theorem for this Newton-like method (1.6) under the γ-condition, using the majorising principle. In Section 2, we introduce the majorising sequences and give some of their properties in presenting our new semilocal convergence theorem where the derivative is assumed to satisfy the γ-condition. In Section 3, some numerical examples are discussed where the conditions of our theorem are satisfied but the hypotheses of other such theorems in Gutiérrez [19], Ezquerro [20], Argyros [21] and Wu [22] are not. Section 4 briefly summarises our results.