East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.221216.250417a

Vol. **7**, No. 3, pp. 508-529
August 2017

A Posteriori Error Estimator for a Weak Galerkin Finite Element Solution of the Stokes Problem

Xiaobo Zheng and Xiaoping Xie*

School of Mathematics, Sichuan University, Chengdu 610064, China.

Received 22 December 2016; Accepted (in revised version) 25 April 2017.

Abstract. A robust residual-based *a posteriori* error estimator is proposed for a weak Galerkin finite element method for the Stokes problem in two and three dimensions. The estimator consists of two terms, where the first term characterises the difference between the L^2-projection of the velocity approximation on the element interfaces and the corresponding numerical trace, and the second is related to the jump of the velocity approximation between the adjacent elements. We show that the estimator is reliable and efficient through two estimates of global upper and global lower bounds, up to two data oscillation terms caused by the source term and the nonhomogeneous Dirichlet boundary condition. The estimator is also robust in the sense that the constant factors in the upper and lower bounds are independent of the viscosity coefficient. Numerical results are provided to verify the theoretical results.

AMS subject classifications: 65N15, 65N30, 76D07

Key words: The Stokes equations, weak Galerkin method, *a posteriori* error estimator.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded polygonal or polyhedral domain. We consider the following generalised Stokes problem: find the velocity $\boldsymbol{u}$ and the pressure p such that

$$\begin{aligned} &-\nu\Delta\boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f} \quad \text{in}\ \Omega\,,\\ &\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \quad \text{in}\ \Omega\,,\\ &\boldsymbol{u} = \boldsymbol{g} \quad \text{on}\ \partial\Omega\,. \end{aligned} \tag{1.1}$$

where $\nu > 0$ denotes the viscosity coefficient, Δ denotes the Laplacian differential operator, $\boldsymbol{f} \in [L^2(\Omega)]^d$ is the body force and $\boldsymbol{g}$ satisfies the compatibility condition $\int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{n} = 0$, with $\boldsymbol{n}$ the unit outward vector normal to the boundary $\partial\Omega$.

These equations describe steady viscous incompressible flow, and the development of reliable and efficient *a posteriori* error estimators for finite element discretisations of this problem has become an active research area in recent decades — cf. Refs. [1–4,7–11,13,14,

*Corresponding author. *Email addresses:* `zhengxiaobosc@yahoo.com` (X. Zheng), `xpxie@scu.edu.cn` (X. Xie)

16, 18, 19] and references therein. Specifically, two *a posteriori* error estimators have been produced for the mini-element based on the residual of the finite element solution and the solution of local problems [18], and related results can be found in Refs. [1,2]; *a posteriori* error estimators were analysed for non-conforming finite element approximations [8,9,19], discontinuous Galerkin methods [14, 16], and for dual mixed finite element methods [3, 10]; some unified framework for *a posteriori* error estimation for the Stokes problem based on H^1-conforming velocity reconstruction and H(div)-conforming locally conservative flux (stress) reconstruction has been provided [11]; and Refs. [4, 7] discuss *a posteriori* error analysis for quasi-Newtonian fluid flows.

A weak Galerkin (WG) finite element method for the Stokes equations (1.1) in the primary velocity-pressure formulation has been proposed [21]. The method uses a P_k/P_{k-1} ($k \geq 1$) discontinuous finite element combination for the velocity and pressure, with the velocity element being enhanced by polynomials of degree $k-1$ on the interface of the finite element partition. The usual gradient and divergence operators are implemented as distributions in properly-defined spaces.

Optimal-order error estimates were established for the corresponding numerical approximation in various norms. Refs. [5, 22] provide another two classes of WG methods for (1.1), and Ref. [23] presents a divergence-free WG method for quasi-Newtonian Stokes flows. Ref. [6] carried out the first *a posteriori* error analysis of WG methods for diffusion equations, where the residual type *a posteriori* error estimator is a combination of the standard conforming Galerkin and mixed finite elements.

Here we develop a residual type *a posteriori* error estimator for the WG method in Ref. [21] for the Stokes problem (1.1) in two and three dimensions. The *a posteriori* error estimator for the velocity error plus the pressure error consists of two terms. The first term characterises the difference between the L^2-projection of the velocity approximation on the element interfaces and the corresponding numerical trace, and the second term is related to the jump of the velocity approximation between the adjacent elements. We show that the estimator is reliable and efficient with two estimates of global upper and global lower bounds, up to two data oscillation terms caused by the source term and the nonhomogeneous Dirichlet boundary condition, and our *a posteriori* estimation is robust with respect to the viscosity coefficient. The main tool of our analysis is the Helmholtz decomposition for tensor fields.

In Section 2, we first provide some notation before proceeding to summarise the WG scheme [21]. We present our *a posteriori* error estimator in Section 3, and discuss its reliability and efficiency. Some relevant numerical results are produced in Section 4, and we make some final remarks in Section 5.

2. Weak Galerkin (WG) Scheme

2.1. Notation

For any bounded domain $D \subset \mathbb{R}^s$ ($s = d, d-1$), let $H^m(D)$ and $H_0^m(D)$ denote the m^{th} order Sobolev spaces on D, and $\|\cdot\|_{m,D}$, $|\cdot|_{m,D}$ the corresponding norm and seminorm, respectively. We use $(\cdot,\cdot)_{m,D}$ to denote the inner product of $H^m(D)$, and in particular $(\cdot,\cdot)_D := (\cdot,\cdot)_{0,D}$. When $D=\Omega$, we abbreviate $\|\cdot\|_m := \|\cdot\|_{m,\Omega}, |\cdot|_m := |\cdot|_{m,\Omega}, (\cdot,\cdot) := (\cdot,\cdot)_\Omega$; and when $D \subset \mathbb{R}^{d-1}$, we replace $\langle\cdot,\cdot\rangle_D$ with $(\cdot,\cdot)_D$. For an integer $k \geq 0$, $P_k(D)$ denotes the set of all polynomials defined on D with degree no greater than k.

Let $\mathscr{T}_h = \cup\{T\}$ be a shape-regular simplicial decomposition of the polyhedral domain Ω with mesh size $h := \max_{T\in\mathscr{T}_h} h_T$ where h_T is the diameter of T; and denote by ε_h^0 and ε_h^∂ the sets of interior and boundary edges/faces of all elements in $\mathscr{T}_h$, respectively, and set $\varepsilon_h := \varepsilon_h^0 \cup \varepsilon_h^\partial$. Moreover, we define $P_k(\mathscr{T}_h) := \{\boldsymbol{v} \in L^2(\Omega) : \boldsymbol{v}|_T \in P_k(T), T \in \mathscr{T}_h\}$, and $\boldsymbol{n}$ now denotes the unit outward normal vector to the boundary ∂T for any $T \in \mathscr{T}_h$. Given a tensor function $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}$ and a vector function $\boldsymbol{v} \in \mathbb{R}^d$, for any two adjacent elements T^+ and $T^- \in \mathscr{T}_h$ that share a common edge/face $e \in \varepsilon_h^0$ (viz. $e = T^+ \cap T^-$) we denote by $\boldsymbol{v}^\pm$ and $\boldsymbol{\tau}^\pm$ the respective functions on the edge/face e taken from the interior of $T^\pm$, and likewise appropriately write $\boldsymbol{n}^\pm$ to denote the unit outward normal vector on e exterior to $T^\pm$. Jumps in quantities along e may be defined as

$$
\begin{aligned}
&[\![\boldsymbol{\tau}\boldsymbol{n}]\!]_e := (\boldsymbol{\tau}\boldsymbol{n})\big|_{T^+\cap\, e} + (\boldsymbol{\tau}\boldsymbol{n})\big|_{T^-\cap\, e}\,,\\
&[\![\boldsymbol{v}]\!]_e := \boldsymbol{v}|_{T^+\cap\, e} - \boldsymbol{v}|_{T^-\cap\, e}\,.
\end{aligned}
\tag{2.1}
$$

We also adopt the usual notation '×' to denote the vector product of two vectors in $\mathbb{R}^d$, such that for $\boldsymbol{v} = (v_1,\cdots,v_d)^t$ and $\boldsymbol{w} = (w_1,\cdots,w_d)^t$ (where t denotes the transpose) we have

$$
\boldsymbol{v}\times\boldsymbol{w} := \begin{cases} v_1w_2 - v_2w_1 & \text{for } d=2\,,\\ (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)^t & \text{for } d=3\,, \end{cases}
\tag{2.2}
$$

and for the vector $\boldsymbol{v} \in \mathbb{R}^d$ and the tensor $\boldsymbol{\tau} = [\tau_1,\cdots,\tau_d]^t \in \mathbb{R}^{d\times d}$ we define

$$
\boldsymbol{v}\times\boldsymbol{\tau} := [\boldsymbol{v}\times\tau_1,\cdots,\boldsymbol{v}\times\tau_d]^t\,.
$$

In addition, we have

$$
\mathbf{curl}\,\boldsymbol{v} := \begin{cases} \begin{bmatrix} -\partial_2 v_1 & \partial_1 v_1\\ -\partial_2 v_2 & \partial_1 v_2 \end{bmatrix} & \text{for } d=2\,,\\ \nabla\times\boldsymbol{v} & \text{for } d=3\,, \end{cases}
$$

where $\nabla = (\partial_1,\cdots,\partial_d)^t$ is the gradient operator; and the Green formula

$$
\langle \boldsymbol{n}\times\boldsymbol{\tau}, \boldsymbol{\theta}\rangle_{\partial D} = (\nabla\times\boldsymbol{\tau}, \boldsymbol{\theta})_D + (\boldsymbol{\tau}, \mathbf{curl}\,\boldsymbol{\theta})_D
$$

where $\boldsymbol{\theta}$ denotes either a vector function in $\mathbb{R}^2$ or a tensor function in $\mathbb{R}^{3\times3}$. We use ∇_h and $\nabla_h\cdot$ to denote the piecewise gradient and divergence operators with respect to the triangulation $\mathscr{T}_h$, respectively. Finally, we also introduce the "broken" Sobolev space $H^m(\mathscr{T}_h) := \{\boldsymbol{v} \in L^2(\Omega) : \boldsymbol{v}|_T \in H^m(T), T \in \mathscr{T}_h\}$, with inner products and norms defined as follows: for any scalar $w, v \in H^m(\mathscr{T}_h)$,

$$
(w,v)_{\mathscr{T}_h} := \sum_{T\in\mathscr{T}_h} (w,v)_T\,, \qquad \|w\|^2_{m,\mathscr{T}_h} := \sum_{T\in\mathscr{T}_h} \|w\|^2_{m,T}\,,
$$

$$\langle w, v\rangle_{\partial T_h} := \sum_{T\in\mathcal{T}_h} \langle w, v\rangle_{\partial T}\,, \quad \|w\|^2_{m,\partial T_h} := \sum_{T\in\mathcal{T}_h} \|w\|^2_{m,\partial T}\,,$$

with analogous definitions for vectors and tensors. Throughout, $X \lesssim Y$ means $X \le CY$, where C is a positive constant independent of the mesh size h and viscosity coefficient ν.

2.2. WG scheme.

For any $e \in \varepsilon_h$, we let $\boldsymbol{Q}_b^k|_e$ be the L^2-projection from $[L^2(e)]^d$ onto $[P_k(e)]^d$, and define discrete weak function spaces for any integer $k \ge 0$ — viz.

$$V_h := \big\{\boldsymbol{v} = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} : \boldsymbol{v}_0|_T \in [P_{k+1}(T)]^d, \boldsymbol{v}_b|_e \in [P_k(e)]^d, \forall T\in\mathcal{T}_h, e \in \varepsilon_h^0, \boldsymbol{v}_b = \boldsymbol{Q}_b^k \boldsymbol{g} \text{ on } \partial\Omega\big\},$$

$$V_h^0 := \big\{\boldsymbol{v} = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} : \boldsymbol{v}_0|_T \in [P_{k+1}(T)]^d, \boldsymbol{v}_b|_e \in [P_k(e)]^d, \forall T\in\mathcal{T}_h, e \in \varepsilon_h^0, \boldsymbol{v}_b = \boldsymbol{0} \text{ on } \partial\Omega\big\},$$

$$W_h := \big\{q \in L_0^2(\Omega) : q|_T \in P_k(T)\,, \forall T \in \mathcal{T}_h\big\}.$$

We follow Ref. [21] in now defining the discrete weak gradient and divergence operators.

Definition 2.1. For any $\boldsymbol{v}_h = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} \in V_h$ or V_h^0, the weak gradient $\boldsymbol{\nabla}_w \boldsymbol{v}_h$ is defined as follows: for any $T \in \mathcal{T}_h$, $(\boldsymbol{\nabla}_w \boldsymbol{v}_h)|_T \in [P_k(T)]^{d\times d}$ satisfies the equation

$$(\boldsymbol{\nabla}_w \boldsymbol{v}_h, \boldsymbol{\tau}_k)_T = \langle \boldsymbol{v}_b, \boldsymbol{\tau}_k \boldsymbol{n}\rangle_{\partial T} - (\boldsymbol{v}_0, \boldsymbol{\nabla}\cdot \boldsymbol{\tau}_k)_T\,, \quad \forall \boldsymbol{\tau}_k \in [P_k(T)]^{d\times d}\,, \tag{2.3}$$

where $\boldsymbol{n}$ denotes the unit outward normal vector along ∂T.

Definition 2.2. For any $\boldsymbol{v}_h = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} \in V_h$ or V_h^0, the weak divergence $\boldsymbol{\nabla}_w\cdot \boldsymbol{v}_h$ is defined as follows: for any $T \in \mathcal{T}_h$, $(\boldsymbol{\nabla}_w\cdot \boldsymbol{v}_h)|_T \in P_k(T)$ satisfies the equation

$$(\boldsymbol{\nabla}_w\cdot \boldsymbol{v}_h, w_k)_T = \langle \boldsymbol{v}_b, w_k \boldsymbol{n}\rangle_{\partial T} - (\boldsymbol{v}_0, \boldsymbol{\nabla} w_k)_T\,, \quad \forall w_k \in P_k(T)\,. \tag{2.4}$$

The weak gradient (WG) finite element method is then as follows. Find $\boldsymbol{u}_h = \{\boldsymbol{u}_0, \boldsymbol{u}_b\} \in V_h$ and $p_h \in W_h$ such that

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h) = (\boldsymbol{f}, \boldsymbol{v}_0)\,, \quad \forall \boldsymbol{v}_h = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} \in V_h^0\,, \tag{2.5}$$

$$b(\boldsymbol{u}_h, q) = 0\,, \ \forall q \in W_h\,, \tag{2.6}$$

where

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) := \nu(\boldsymbol{\nabla}_w \boldsymbol{u}_h, \boldsymbol{\nabla}_w \boldsymbol{v}_h)_{\mathcal{T}_h} + \nu \sum_{T\in\mathcal{T}_h} h_T^{-1} \langle \boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b, \boldsymbol{Q}_b^k \boldsymbol{v}_0 - \boldsymbol{v}_b\rangle_{\partial T}\,,$$

$$b(\boldsymbol{v}_h, q) := (\boldsymbol{\nabla}_w\cdot \boldsymbol{v}_h, q)_{\mathcal{T}_h}\,.$$

As shown in Ref. [21], the solution $(\boldsymbol{u}_h, p_h)$ of Eqs. (2.5) and (2.6) exists and is unique, and the following theorem on *a priori* error estimates holds.

Theorem 2.1. *Assume that* $(\boldsymbol{u}, p) \in [H^{k+2}(\Omega)]^d \times H^{k+1}(\Omega)$ *and* $(\boldsymbol{u}_h, p_h) \in V_h \times W_h$ *are the solution of the Stokes problem* (1.1) *and the discrete solution of the weak Galerkin scheme* (2.5)-(2.6)*, respectively. Then*

$$\|\boldsymbol{u} - \boldsymbol{u}_0\|_0 \le Ch^{k+2}\big(|\boldsymbol{u}|_{k+2} + |p|_{k+1}\big)\,,$$

$$\|p - p_h\|_0 \le Ch^{k+1}\big(|\boldsymbol{u}|_{k+2} + |p|_{k+1}\big)\,,$$

where the positive constant C is independent of the mesh size h.

3. *A Posteriori* Error Estimation

For the weak solution $(\boldsymbol{u},p)$ of the Stokes problem (1.1) and the discrete solution $(\boldsymbol{u}_h,p_h)$ with $\boldsymbol{u}_h=\{\boldsymbol{u}_0,\boldsymbol{u}_b\}$ of the WG scheme (2.5)-(2.6), we define the error e_h and the residual-based *a posteriori* error estimator η_h such that

$$e_h^2=\nu\|\nabla\boldsymbol{u}-\nabla_w\boldsymbol{u}_h\|_0^2+\nu\|\nabla\boldsymbol{u}-\nabla_h\boldsymbol{u}_0\|_0^2+\nu^{-1}\|p-p_h\|_0^2\,, \tag{3.1}$$

$$\eta_h^2=\eta_{b,h}^2+\eta_{J,h}^2\,, \tag{3.2}$$

where

$$\eta_{b,h}^2:=\nu\sum_{T\in\mathscr{T}_h}h_T^{-1}\|\boldsymbol{Q}_b^k\boldsymbol{u}_0-\boldsymbol{u}_b\|_{0,\partial T}^2\,,\qquad \eta_{J,h}^2:=\nu\sum_{e\in\varepsilon_h}h_e^{-1}\big\|\mathbf{J}_e(\boldsymbol{u}_0)\big\|_{0,e}^2\,,$$

$$\mathbf{J}_e(\boldsymbol{u}_0):=\begin{cases}[\![\boldsymbol{u}_0]\!]_e & \text{for } e\in\varepsilon_h^0,\\ \boldsymbol{u}_0-\boldsymbol{g} & \text{for } e\in\varepsilon_h^\partial.\end{cases}$$

We also identify oscillation terms in both the source term $\boldsymbol{f}$ and the nonhomogeneous Dirichlet boundary data $\boldsymbol{g}$. Let $\boldsymbol{Q}_0^{k+1}$ be the L^2-projection from $[L^2(\Omega)]^d$ onto $[P_{k+1}(\mathscr{T}_h)]^d$, and denote $\boldsymbol{f}_h:=\boldsymbol{Q}_0^{k+1}\boldsymbol{f}$. Then we define the oscillation $\mathrm{osc}(\boldsymbol{f},\mathscr{T}_h)$ of $\boldsymbol{f}$ such that

$$\mathrm{osc}^2(\boldsymbol{f},\mathscr{T}_h):=\nu^{-1}\sum_{T\in\mathscr{T}_h}h_T^2\|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,T}^2\,; \tag{3.3}$$

and assuming that $\boldsymbol{g}\in H^1(\varepsilon_h^\partial)$, we define the oscillation $\mathrm{osc}(\boldsymbol{g},\varepsilon_h^\partial)$ of $\boldsymbol{g}$ such that

$$\mathrm{osc}^2(\boldsymbol{g},\varepsilon_h^\partial)=\nu\sum_{e\in\varepsilon_h^\partial}h_e\big\|(\boldsymbol{n}\times\nabla)(\boldsymbol{g}-\mathscr{P}^{k+1}\boldsymbol{g})\big\|_{0,e}^2 \tag{3.4}$$

where $\mathscr{P}^{k+1}\boldsymbol{g}\in[H^1(\partial\Omega)\cap\ P_{k+1}(\varepsilon_h^\partial)]^d$ is an approximation of $\boldsymbol{g}$ constructed by first calculating $\boldsymbol{Q}_b^{k+1}\boldsymbol{g}$, and then average the values of $\boldsymbol{Q}_b^{k+1}\boldsymbol{g}$ that share the same node to get the continuous piecewise polynomial $\mathscr{P}^{k+1}\boldsymbol{g}$ of degree $k+1$ or less.

Remark 3.1. We note that $\boldsymbol{g}$ is assumed to be a continuous piecewise linear polynomial vector in Refs. [2, 12], so no data oscillation appears in the *a posteriori* error estimation. (From Eq. (3.4), $\mathrm{osc}(\boldsymbol{g},\varepsilon_h^\partial)$ vanishes if $\boldsymbol{g}\in[H^1(\partial\Omega)\cap P_{k+1}(\varepsilon_h^\partial)]^d$.)

3.1. Reliability

To assess the reliability of the error estimator η_h, we need some auxiliary results for the WG solution $(\boldsymbol{u}_h,p_h)\in V_h\times W_h$, assembled in Lemmas 3.1-3.3 below.

Lemma 3.1. *For*

$$S_1:=\big\{\boldsymbol{v}\in[H_0^1(\Omega)]^d;\ \boldsymbol{v}|_T\in[P_1(T)]^d,\ \forall T\in\mathscr{T}_h\big\}\,, \tag{3.5}$$

we have

$$\nu(\nabla_w\boldsymbol{u}_h,\nabla\tilde{\boldsymbol{v}})_{\mathscr{T}_h}-(p_h,\nabla\cdot\tilde{\boldsymbol{v}})_{\mathscr{T}_h}=(\boldsymbol{f},\tilde{\boldsymbol{v}})_{\mathscr{T}_h},\quad \forall\tilde{\boldsymbol{v}}\in S_1\,. \tag{3.6}$$

Proof. From Eqs. (2.3)-(2.4), we can rewrite Eq. (2.5) as

$$\begin{aligned}&\nu(\nabla_w \boldsymbol{u}_h, \nabla_h \boldsymbol{v}_0)_{\mathscr{T}_h} - (p_h, \nabla_h \cdot \boldsymbol{v}_0)_{\mathscr{T}_h} + \big\langle(-\nu\nabla_w \boldsymbol{u}_h + p_h \mathbf{I})\boldsymbol{n}, \boldsymbol{Q}_b^k \boldsymbol{v}_0 - \boldsymbol{v}_b\big\rangle_{\partial\mathscr{T}_h} \\ &\quad + \nu\big\langle h_T^{-1}(\boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b), \boldsymbol{Q}_b^k \boldsymbol{v}_0 - \boldsymbol{v}_b\big\rangle_{\partial\mathscr{T}_h} = (\boldsymbol{f}, \boldsymbol{v}_0)_{\mathscr{T}_h}, \quad \forall \boldsymbol{v}_h = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} \in V_h^0, \end{aligned} \tag{3.7}$$

where $\mathbf{I}$ denotes the appropriate unit dyadic (with representation the $d \times d$ unit matrix). For any $\tilde{\boldsymbol{v}} \in S_1$, we take $\boldsymbol{v}_0 = \tilde{\boldsymbol{v}}$, $\boldsymbol{v}_b = \boldsymbol{Q}_b^k \tilde{\boldsymbol{v}}$. Since $\tilde{\boldsymbol{v}}$ is continuous across each edge/face, we have $\boldsymbol{Q}_b^k \boldsymbol{v}_0 = \boldsymbol{v}_b$, hence Eq. (3.6) holds. □

Let G be an element, an edge or a node of $\mathscr{T}_h$, and denote a patch of G by

$$w_G := \{T \in \mathscr{T}_h;\ T \cap G = G\}. \tag{3.8}$$

Lemma 3.2. *For any $T \in \mathscr{T}_h$ and $e \in \partial T \cap \varepsilon_h$, the following estimates hold:*

$$\|\nabla_h \boldsymbol{u}_0 - \nabla_w \boldsymbol{u}_h\|_{0,T} \lesssim h_T^{-\frac{1}{2}} \|\boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b\|_{0,\partial T}, \tag{3.9}$$

$$h_T \|\nabla \times \nabla_w \boldsymbol{u}_h\|_{0,T} \lesssim h_T^{-\frac{1}{2}} \|\boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b\|_{0,\partial T}, \tag{3.10}$$

$$h_T \big\|\boldsymbol{f}_h + \nu\nabla_h \cdot (\nabla_w \boldsymbol{u}_h) - \nabla_h p_h\big\|_{0,T} \lesssim \nu h_T^{-\frac{1}{2}} \|\boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b\|_{0,\partial T}, \tag{3.11}$$

$$h_e^{\frac{1}{2}} \big\|[\![(\nu\nabla_w \boldsymbol{u}_h - p_h \mathbf{I})\boldsymbol{n}]\!]\big\|_{0,e} \lesssim \nu \sum_{T \in w_e} h_T^{-\frac{1}{2}} \|\boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b\|_{0,e\cap\partial T}. \tag{3.12}$$

Proof. Firstly, we prove the estimate (3.9). From Eq. (2.3), integration by parts, the trace inequality and the inverse inequality, with $\boldsymbol{\tau}_k = \nabla_h \boldsymbol{u}_0 - \nabla_w \boldsymbol{u}_h \in [P_k(T)]^{d\times d}$ we have

$$\begin{aligned}&(\nabla_h \boldsymbol{u}_0 - \nabla_w \boldsymbol{u}_h, \boldsymbol{\tau}_k) \\ &= (\nabla_h \boldsymbol{u}_0, \boldsymbol{\tau}_k)_T - (\nabla_w \boldsymbol{u}_h, \boldsymbol{\tau}_k)_T \\ &= (\nabla_h \boldsymbol{u}_0, \boldsymbol{\tau}_k)_T - \langle \boldsymbol{u}_b, \boldsymbol{\tau}_k \boldsymbol{n}\rangle_{\partial T} + (\boldsymbol{u}_0, \nabla\cdot \boldsymbol{\tau}_k)_T \\ &= \langle \boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b, \boldsymbol{\tau}_k \boldsymbol{n}\rangle_{\partial T} \\ &\lesssim h_T^{-\frac{1}{2}} \|\boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b\|_{0,\partial T} \|\boldsymbol{\tau}_k\|_{0,T}\end{aligned}$$

— i.e. estimate (3.9). Similarly, to prove estimate (3.10) we take $\boldsymbol{\tau}_{k-1} = \nabla \times \nabla_w \boldsymbol{u}_h$ to obtain

$$\begin{aligned}&(\nabla \times \nabla_w \boldsymbol{u}_h, \boldsymbol{\tau}_{k-1})_T \\ &= \langle \boldsymbol{n} \times \nabla_w \boldsymbol{u}_h, \boldsymbol{\tau}_{k-1}\rangle_{\partial T} - (\nabla_w \boldsymbol{u}_h, \mathbf{curl}\ \boldsymbol{\tau}_{k-1})_T \\ &= \langle \boldsymbol{n} \times \nabla_w \boldsymbol{u}_h, \boldsymbol{\tau}_{k-1}\rangle_{\partial T} - \langle \boldsymbol{u}_b, \mathbf{curl}\ \boldsymbol{\tau}_{k-1} \boldsymbol{n}\rangle_{\partial T} + (\boldsymbol{u}_0, \nabla\cdot \mathbf{curl}\ \boldsymbol{\tau}_{k-1})_T \\ &= \langle \boldsymbol{n} \times \nabla_w \boldsymbol{u}_h, \boldsymbol{\tau}_{k-1}\rangle_{\partial T} + \langle \boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b, \mathbf{curl}\ \boldsymbol{\tau}_{k-1} \boldsymbol{n}\rangle_{\partial T} - (\nabla_h \boldsymbol{u}_0, \mathbf{curl}\ \boldsymbol{\tau}_{k-1})_T \\ &= \big\langle \boldsymbol{n} \times (\nabla_w \boldsymbol{u}_h - \nabla_h \boldsymbol{u}_0), \boldsymbol{\tau}_{k-1}\big\rangle_{\partial T} + \langle \boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b, \mathbf{curl}\ \boldsymbol{\tau}_{k-1} \boldsymbol{n}\rangle_{\partial T} + (\nabla \times \nabla_h \boldsymbol{u}_0, \boldsymbol{\tau}_{k-1})_T \\ &\lesssim h_T^{-1} \|\nabla_w \boldsymbol{u}_h - \nabla_h \boldsymbol{u}_0\|_{0,T} \|\boldsymbol{\tau}_{k-1}\|_{0,T} + h_T^{-\frac{3}{2}} \|\boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b\|_{0,\partial T} \|\boldsymbol{\tau}_{k-1}\|_{0,T},\end{aligned}$$

which combined with (3.9) implies the estimate (3.10). For the estimate (3.11), with $\boldsymbol{v}_h = \{\boldsymbol{v}_0, \mathbf{0}\}$ in (3.7) combined with $\boldsymbol{v}_0 = \boldsymbol{f}_h + \nu\nabla_h \cdot (\nabla_w \boldsymbol{u}_h) - \nabla_h p_h$ we get

$$\begin{aligned}\big(\boldsymbol{f}_h + \nu\nabla_h\cdot(\nabla_w\boldsymbol{u}_h) - \nabla_h p_h, \boldsymbol{v}_0\big)_T &= \nu h_T^{-1}\langle \boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b, \boldsymbol{v}_0\rangle_{\partial T} \\ &\lesssim \nu h_T^{-\frac{3}{2}} \|\boldsymbol{Q}_b^k\boldsymbol{u}_0 - \boldsymbol{u}_b\|_{0,\partial T}\cdot\|\boldsymbol{v}_0\|_{0,T}\,.\end{aligned}$$

Finally, in order to prove the estimate (3.12), on any edge/face $e\in\varepsilon_h$ we take $\boldsymbol{v}_0 = 0$ with $\boldsymbol{v}_b = [\![(\nabla_w\boldsymbol{u}_h - p_h\mathbf{I})\boldsymbol{n}]\!]_e$ in Eq. (3.7) such that

$$\begin{aligned}\big\langle [\![(\nu\nabla_w\boldsymbol{u}_h - p_h\mathbf{I})\boldsymbol{n}]\!], \boldsymbol{v}_b\big\rangle_e &= \nu\sum_{T\in w_e} h_T^{-1}\langle \boldsymbol{Q}_b^k\boldsymbol{u}_0 - \boldsymbol{u}_b, \boldsymbol{v}_b\rangle_{e\cap\partial T} \\ &\lesssim \nu\|\boldsymbol{v}_b\|_{0,e}\sum_{T\in w_e} h_T^{-1}\|\boldsymbol{Q}_b^k\boldsymbol{u}_0 - \boldsymbol{u}_b\|_{0,e\cap\partial T}\,,\end{aligned}$$

so the estimate (3.12) follows since $h_T \lesssim h_e$ due to the shape regular condition. □

Lemma 3.3. *For the error term $\boldsymbol{e}_u := \nabla\boldsymbol{u} - \nabla_w\boldsymbol{u}_h$, the following decomposition holds:*

$$\boldsymbol{e}_u = \nabla\boldsymbol{\psi} - \nu^{-1}r\mathbf{I} + \nu^{-1}\mathbf{curl}\,\boldsymbol{\phi} \tag{3.13}$$

with $\boldsymbol{\psi}\in[H_0^1(\Omega)]^d$ and $r\in L_0^2(\Omega)$, and $\boldsymbol{\phi}\in[H^1(\Omega)\cap L_0^2(\Omega)]^2$ and $\boldsymbol{\phi}\in[H^1(\Omega)\cap L_0^2(\Omega)]^{3\times3}$ when $d=2$ and $d=3$ respectively, where $\nabla\cdot\boldsymbol{\psi}=0$ and

$$\nu\|\nabla\boldsymbol{\psi}\|_0 + \|r\|_0 + \|\nabla\boldsymbol{\phi}\|_0 \lesssim \nu\|\boldsymbol{e}_u\|_0\,. \tag{3.14}$$

Proof. We follow Ref. [8] to derive the decomposition (3.13). If $\boldsymbol{\psi}\in[H_0^1(\Omega)]^d$, $r\in L_0^2(\Omega)$ is the solution of the Stokes equations with the right-hand side $-\nu\nabla\cdot\boldsymbol{e}_u\in[H^{-1}(\Omega)]^d$, then

$$-\nu\Delta\boldsymbol{\psi} + \nabla r = -\nu\nabla\cdot\boldsymbol{e}_u\,, \tag{3.15}$$

$$\nabla\cdot\boldsymbol{\psi} = 0\,, \tag{3.16}$$

and we have the standard estimate

$$\nu\|\nabla\boldsymbol{\psi}\|_0 + \|r\|_0 \lesssim \nu\|\boldsymbol{e}_u\|_0\,. \tag{3.17}$$

On the other hand, we can rewrite Eq. (3.15) as

$$\nabla\cdot(\nu\nabla\boldsymbol{\psi} - r\mathbf{I} - \nu\boldsymbol{e}_u) = 0\,,$$

hence there exists a $\boldsymbol{\phi}\in[H^1(\Omega)\cap L_0^2(\Omega)]^2$ or $[H^1(\Omega)\cap L_0^2(\Omega)]^{3\times3}$ such that Eq. (3.13) holds. The estimate (3.14) then follows from Eq. (3.13) and the estimate (3.17). □

Now we are ready to state the following reliability result for the error estimator η_h.

Theorem 3.1 (Upper bound). *Let $(\boldsymbol{u},p)$ be the solution of the Stokes problem* (1.1)*, and $(\boldsymbol{u}_h,p_h)$ the solution of the weak Galerkin scheme Eqs.* (2.5)-(2.6)*. Then the following reliability estimate holds:*

$$e_h^2 \lesssim \eta_h^2 + osc^2(\boldsymbol{f},\mathscr{T}_h) + osc^2(\boldsymbol{g},\varepsilon_h^{\partial})\,. \tag{3.18}$$

Proof. Recalling $e_h^2 = \nu\|\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h\|_0^2 + \nu\|\nabla \boldsymbol{u} - \nabla_h \boldsymbol{u}_0\|_0^2 + \nu^{-1}\|p - p_h\|_0^2$, we prove the result in three steps.

Step 1. Estimate the error term $\nu\|\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h\|_0^2$.

From Lemma 3.3, $\boldsymbol{e}_u := \nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h = \nabla \boldsymbol{\psi} - \nu^{-1} r\mathbf{I} + \nu^{-1}\mathbf{curl}\ \boldsymbol{\phi}$ such that

$$\nu\|\boldsymbol{e}_u\|_0^2 = \nu(\boldsymbol{e}_u, \nabla\boldsymbol{\psi})_{\mathscr{T}_h} - (\boldsymbol{e}_u, r\mathbf{I})_{\mathscr{T}_h} + (\boldsymbol{e}_u, \mathbf{curl}\ \boldsymbol{\phi})_{\mathscr{T}_h}\,. \tag{3.19}$$

We have

$$(\boldsymbol{e}_u, r\mathbf{I})_{\mathscr{T}_h} = (\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h, r\mathbf{I})_{\mathscr{T}_h} = 0\,, \tag{3.20}$$

since obviously $(\nabla \boldsymbol{u}, r\mathbf{I})_{\mathscr{T}_h} = (\nabla\cdot \boldsymbol{u}, r) = 0$ and from Definitions 2.1, 2.2 and Eq. (2.6)

$$\begin{aligned}(\nabla_w \boldsymbol{u}_h, r\mathbf{I})_{\mathscr{T}_h} &= (\nabla_w \boldsymbol{u}_h, \mathscr{J}^k r\mathbf{I})_{\mathscr{T}_h}\\ &= (\nabla_w\cdot \boldsymbol{u}_h, \mathscr{J}^k r)_{\mathscr{T}_h}\\ &= 0\,,\end{aligned}$$

on letting $\mathscr{J}^k$ be the L^2-projection from $L^2(\Omega)$ onto $P_k(\mathscr{T}_h)$ and recalling that $\mathscr{J}^k r \in W_h$ for $r \in L_0^2(\Omega)$. To estimate the term $\nu(\boldsymbol{e}_u, \nabla\boldsymbol{\psi})_{\mathscr{T}_h}$ in Eq. (3.19), from integration by parts and noting that $\boldsymbol{\psi} \in [H_0^1(\Omega)]^d$ with $\nabla\cdot\boldsymbol{\psi} = 0$, the first equation of (1.1), and Eq. (3.6), we have

$$\begin{aligned}\nu(e_u, \nabla\boldsymbol{\psi})_{\mathscr{T}_h} =& -\nu(\Delta \boldsymbol{u}, \boldsymbol{\psi}) - \nu(\nabla_w \boldsymbol{u}_h, \nabla\boldsymbol{\psi})_{\mathscr{T}_h}\\ =&(\boldsymbol{f} - \nabla p, \boldsymbol{\psi}) - \nu(\nabla_w \boldsymbol{u}_h, \nabla\boldsymbol{\psi})_{\mathscr{T}_h}\\ =&(\boldsymbol{f}, \boldsymbol{\psi}) - \nu(\nabla_w \boldsymbol{u}_h, \nabla\boldsymbol{\psi})_{\mathscr{T}_h}\\ =&(\boldsymbol{f}, \boldsymbol{\psi} - \boldsymbol{\psi}_h)_{\mathscr{T}_h} - \nu\big(\nabla_w \boldsymbol{u}_h, \nabla(\boldsymbol{\psi} - \boldsymbol{\psi}_h)\big)_{\mathscr{T}_h} + (\boldsymbol{f}, \boldsymbol{\psi}_h)_{\mathscr{T}_h} - \nu(\nabla_w \boldsymbol{u}_h, \nabla\boldsymbol{\psi}_h)_{\mathscr{T}_h}\\ =&\big(\boldsymbol{f} + \nu\nabla_h\cdot(\nabla_w \boldsymbol{u}_h), \boldsymbol{\psi} - \boldsymbol{\psi}_h\big)_{\mathscr{T}_h} - \big\langle(\nu\nabla_w \boldsymbol{u}_h)\boldsymbol{n}, \boldsymbol{\psi} - \boldsymbol{\psi}_h\big\rangle_{\partial\mathscr{T}_h}\\ &+ \big(p_h, \nabla\cdot(\boldsymbol{\psi} - \boldsymbol{\psi}_h)\big)_{\mathscr{T}_h}\\ =&\big(\boldsymbol{f}_h + \nu\nabla_h\cdot(\nabla_w \boldsymbol{u}_h) - \nabla_h p_h, \boldsymbol{\psi} - \boldsymbol{\psi}_h\big)_{\mathscr{T}_h} + (\boldsymbol{f} - \boldsymbol{f}_h, \boldsymbol{\psi} - \boldsymbol{\psi}_h)_{\mathscr{T}_h}\\ &+ \sum_{e\in\varepsilon_h^0} \big\langle [\![(-\nu\nabla_w \boldsymbol{u}_h + p_h\mathbf{I})\boldsymbol{n}]\!], \boldsymbol{\psi} - \boldsymbol{\psi}_h\big\rangle_e\,,\end{aligned}$$

where $\boldsymbol{\psi}_h \in S_1$ as in (3.5) is an interpolation of $\boldsymbol{\psi}$ satisfying the following estimates [6]: for any $T \in \mathscr{T}_h$ and $e \in \varepsilon_h$,

$$\begin{aligned}\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{0,T} &\lesssim h_T\|\nabla\boldsymbol{\psi}\|_{0,w_T}\,,\\ \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{0,e} &\lesssim h_e^{\frac{1}{2}}\|\nabla\boldsymbol{\psi}\|_{0,w_e}\,,\end{aligned} \tag{3.21}$$

with w_T, w_e as defined by (3.8). Consequently, from Eq. (3.1) and the estimates (3.11), (3.12), (3.17) and (3.21) we obtain

$$\begin{aligned}\nu(e_u,\nabla\psi) &\lesssim \nu^{-\frac{1}{2}}\Bigg[\bigg(\sum_{T\in\mathcal{T}_h} h_T^2\|\boldsymbol{f}_h+\nu\nabla_h\cdot(\nabla_w\boldsymbol{u}_h)-\nabla_h p_h\|_{0,T}^2\bigg)^{1/2}+\bigg(\sum_{T\in\mathcal{T}_h} h_T^2\|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,T}^2\bigg)^{1/2}\\ &\qquad+\bigg(\sum_{e\in\varepsilon_h^0} h_e^2\|[\![(-\nu\nabla_w\boldsymbol{u}_h+p_h\mathbf{I})\boldsymbol{n}]\!]\|_{0,e}^2\bigg)^{1/2}\Bigg]\nu^{\frac{1}{2}}\|\nabla\psi\|_0\\ &\lesssim\Bigg[\bigg(\nu\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\boldsymbol{Q}_b^k\boldsymbol{u}_0-\boldsymbol{u}_b\|_{0,\partial T}^2\bigg)^{1/2}+\bigg(\nu^{-1}\sum_{T\in\mathcal{T}_h} h_T^2\|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,T}^2\bigg)^{1/2}\Bigg]\nu^{\frac{1}{2}}\|\nabla\psi\|_0\\ &\lesssim\big[\eta_h^2+\mathrm{osc}^2(\boldsymbol{f},\mathcal{T}_h)\big]^{1/2}\,\nu^{\frac{1}{2}}\|e_u\|_0\,. \end{aligned}\tag{3.22}$$

Further, to estimate the term $(\boldsymbol{e}_u,\mathbf{curl}\,\boldsymbol{\phi})_{\mathcal{T}_h}$ in (3.19) we let $\boldsymbol{\phi}_h$ be an interpolant of $\boldsymbol{\phi}$ with $\boldsymbol{\phi}_h\in[H^1(\Omega)\cap\ P_1(\mathcal{T}_h)]^2$ when $d=2$ or $\boldsymbol{\phi}_h\in[H^1(\Omega)\cap\ P_1(\mathcal{T}_h)]^{3\times3}$ when $d=3$, such that

$$\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{0,T}\lesssim h_T\|\nabla\boldsymbol{\phi}\|_{0,w_T}\,,\quad \|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{0,e}\lesssim h_e^{\frac{1}{2}}\|\nabla\boldsymbol{\phi}\|_{0,w_e}\,,\ \ \forall T\in\mathcal{T}_h, e\in\varepsilon_h\,. \tag{3.23}$$

Since obviously $\mathbf{curl}\,\boldsymbol{\phi}_h\in[H(div,\Omega)]^d:=\{\boldsymbol{w}\in[L^2(\Omega)]^d,\nabla\cdot\boldsymbol{w}\in[L^2(\Omega)]^d\}$, on using integration by parts and the property of the L^2−projection $\boldsymbol{Q}_b^k$ we have

$$\begin{aligned}(\boldsymbol{e}_u,\mathbf{curl}\,\boldsymbol{\phi}_h)_{\mathcal{T}_h}&=\sum_{T\in\mathcal{T}_h}(\nabla\boldsymbol{u}-\nabla_w\boldsymbol{u}_h,\mathbf{curl}\,\boldsymbol{\phi}_h)_T\\ &=\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{u}-\boldsymbol{u}_b,\mathbf{curl}\,\boldsymbol{\phi}_h\boldsymbol{n}\rangle_{\partial T}\\ &=\sum_{e\in\varepsilon_h^\partial}\langle\boldsymbol{g}-\boldsymbol{Q}_b^k\boldsymbol{g},\mathbf{curl}\,\boldsymbol{\phi}_h\boldsymbol{n}\rangle_e=0\,.\end{aligned}\tag{3.24}$$

For the discrete solution $\boldsymbol{u}_h=\{\boldsymbol{u}_0,\boldsymbol{u}_b\}$, let $\boldsymbol{u}_h^*$ be an approximation to $\boldsymbol{u}_0$ with

$$\boldsymbol{u}_h^*\in\big[P_{k+1}(\mathcal{T}_h)\big]^d\cap\big[H^1(\Omega)\big]^d,\qquad \boldsymbol{u}_h^*|_{\partial\Omega}=\mathscr{P}^{k+1}\boldsymbol{g}\,. \tag{3.25}$$

We can construct $\boldsymbol{u}_h^*$ on assuming that $a_{i,T}$ is the i^{th} Lagrange interpolate node in $T\in\mathcal{T}_h$, and denoting $S_h(a_{i,T}):=\{T'\in\mathcal{T}_h:\ a_{i,T}\in T'\}$. Thus if $N_h(a_{i,T})$ is the number of the elements of $S_h(a_{i,T})$, then the value of $\boldsymbol{u}_h^*$ at the node $a_{i,T}$ is defined by

$$\boldsymbol{u}_h^*(a_{i,T})=\begin{cases}\frac{1}{N_h(a_{i,T})}\sum\limits_{T'\in S_h(a_{i,T})}\boldsymbol{u}_0|_{T'}(a_{i,T}) & \text{for } a_{i,T}\notin\partial\Omega\,,\\ \mathscr{P}^{k+1}\boldsymbol{g}(a_{i,T}) & \text{for } a_{i,T}\in\partial\Omega\,.\end{cases}\tag{3.26}$$

From Ref. [17], we have that

$$\sum_{T\in\mathcal{T}_h} |\boldsymbol{u}_0-\boldsymbol{u}_h^*|_{s,T}^2 \lesssim \sum_{e\in\varepsilon_h} h_e^{1-2s}\|\mathbf{J}_e(\boldsymbol{u}_0)\|_{0,e}^2\,, \quad s=0,1\,, \tag{3.27}$$

and from Eqs. (3.23) that

$$\begin{aligned}\big\langle \boldsymbol{n}\times(\nabla\boldsymbol{u}-\nabla\boldsymbol{u}_h^*),\boldsymbol{\phi}-\boldsymbol{\phi}_h\big\rangle_{\partial\mathcal{T}_h} &= \sum_{e\in\varepsilon_h^\partial}\big\langle(\boldsymbol{n}\times\nabla)(\boldsymbol{g}-\mathcal{P}^{k+1}\boldsymbol{g}),\boldsymbol{\phi}-\boldsymbol{\phi}_h\big\rangle_e\\ &\lesssim \nu^{-\frac12}\|\nabla\boldsymbol{\phi}\|_0\cdot \operatorname{osc}(\boldsymbol{g},\varepsilon_h^\partial)\,,\end{aligned}$$

so from the estimates (3.10), (3.14) and (3.27) combined with Eq. (3.24) we obtain

$$\begin{aligned}(\boldsymbol{e}_u,\mathbf{curl}\,\boldsymbol{\phi})_{\mathcal{T}_h} &= \big(\boldsymbol{e}_u,\mathbf{curl}\,(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\big)_{\mathcal{T}_h}\\ &= \langle\boldsymbol{n}\times\boldsymbol{e}_u,\boldsymbol{\phi}-\boldsymbol{\phi}_h\rangle_{\partial\mathcal{T}_h} - (\nabla\times\boldsymbol{e}_u,\boldsymbol{\phi}-\boldsymbol{\phi}_h)_{\mathcal{T}_h}\\ &= \big\langle\boldsymbol{n}\times(\nabla\boldsymbol{u}_h^*-\nabla_h\boldsymbol{u}_0),\boldsymbol{\phi}-\boldsymbol{\phi}_h\big\rangle_{\partial\mathcal{T}_h} + \big\langle\boldsymbol{n}\times(\nabla\boldsymbol{u}-\nabla\boldsymbol{u}_h^*),\boldsymbol{\phi}-\boldsymbol{\phi}_h\big\rangle_{\partial\mathcal{T}_h}\\ &\quad + \big\langle\boldsymbol{n}\times(\nabla_h\boldsymbol{u}_0-\nabla_w\boldsymbol{u}_h),\boldsymbol{\phi}-\boldsymbol{\phi}_h\big\rangle_{\partial\mathcal{T}_h} + (\nabla\times\nabla_w\boldsymbol{u}_h,\boldsymbol{\phi}-\boldsymbol{\phi}_h)_{\mathcal{T}_h}\\ &\lesssim \nu^{-\frac12}\|\nabla\boldsymbol{\phi}\|\Bigg[\nu^{\frac12}\|\nabla\boldsymbol{u}_h^*-\nabla_h\boldsymbol{u}_0\|_0 + \operatorname{osc}(\boldsymbol{g},\varepsilon_h^\partial)\\ &\quad + \nu^{\frac12}\|\nabla_h\boldsymbol{u}_0-\nabla_w\boldsymbol{u}_h\|_0 + \Bigg(\sum_{T\in\mathcal{T}_h}\nu h_T^2\|\nabla\times\nabla_w\boldsymbol{u}_h\|_{0,T}^2\Bigg)^{1/2}\Bigg]\\ &\lesssim \big(\eta_h+\operatorname{osc}(\boldsymbol{g},\varepsilon_h^\partial)\big)\,\nu^{\frac12}\|\boldsymbol{e}_u\|_0\,.\end{aligned} \tag{3.28}$$

Consequently, combining Eqs. (3.19) and (3.20) with the estimates (3.22) and (3.28) we obtain

$$\begin{aligned}\nu\|\nabla\boldsymbol{u}-\nabla_w\boldsymbol{u}_h\|_0^2 &= \nu\|\boldsymbol{e}_u\|_0^2\\ &\lesssim \eta_h^2+\operatorname{osc}^2(\boldsymbol{f},\mathcal{T}_h)+\operatorname{osc}^2(\boldsymbol{g},\varepsilon_h^\partial)\,.\end{aligned} \tag{3.29}$$

Step 2. We use the triangle inequality and the estimate (3.9) to get

$$\begin{aligned}\nu\|\nabla\boldsymbol{u}-\nabla_h\boldsymbol{u}_0\|_0^2 &\lesssim \nu\|\nabla\boldsymbol{u}-\nabla_w\boldsymbol{u}_h\|_0^2+\nu\|\nabla_w\boldsymbol{u}_h-\nabla_h\boldsymbol{u}_0\|_0^2\\ &\lesssim \eta_h^2+\operatorname{osc}^2(\boldsymbol{f},\mathcal{T}_h)+\operatorname{osc}^2(\boldsymbol{g},\varepsilon_h^\partial)\,.\end{aligned} \tag{3.30}$$

Step 3. Estimate the remaining term of e_h^2, $\nu^{-1}\|p-p_h\|_0^2$.

On the one hand, since $p-p_h\in L_0^2(\Omega)$ the inf-sup condition indicates that

$$\|p-p_h\|_0 \lesssim \sup_{\boldsymbol{v}\in[H_0^1(\Omega)]^d/\{\mathbf{0}\}} \frac{(p-p_h,\nabla\cdot\boldsymbol{v})}{\|\nabla\boldsymbol{v}\|_0}\,.$$

On the other hand, for any $\boldsymbol{v} \in [H_0^1(\Omega)]^d$ let $\tilde{\boldsymbol{v}}_h \in S_1$ be an interpolant of $\boldsymbol{v}$ with the same approximation properties (3.21) as $\boldsymbol{\psi}_h$. It follows that for the Stokes problem (1.1) with Eqs. (3.6) and (3.29) and the estimates (3.11) and (3.12), on integration by parts we have

$$\begin{aligned}
(p-p_h, \nabla\cdot \boldsymbol{v})_{\mathcal{T}_h} =& \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{\mathcal{T}_h} - (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h} - (p_h, \nabla\cdot \boldsymbol{v})_{\mathcal{T}_h} \\
=& \nu(\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h, \nabla \boldsymbol{v})_{\mathcal{T}_h} - (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h} + \nu(\nabla_w \boldsymbol{u}_h, \nabla \boldsymbol{v})_{\mathcal{T}_h} - (p_h, \nabla\cdot \boldsymbol{v})_{\mathcal{T}_h} \\
=& \nu(\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h, \nabla \boldsymbol{v})_{\mathcal{T}_h} - (\boldsymbol{f}, \boldsymbol{v} - \tilde{\boldsymbol{v}}_h)_{\mathcal{T}_h} + \nu\big(\nabla_w \boldsymbol{u}_h, \nabla(\boldsymbol{v} - \tilde{\boldsymbol{v}}_h)\big)_{\mathcal{T}_h} \\
& - (p_h, \nabla\cdot(\boldsymbol{v} - \tilde{\boldsymbol{v}}_h))_{\mathcal{T}_h} \\
=& \nu(\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h, \nabla \boldsymbol{v})_{\mathcal{T}_h} - \big(\boldsymbol{f}_h + \nu\nabla_h\cdot(\nabla_w \boldsymbol{u}_h)_{\mathcal{T}_h} - \nabla_h p_h, \boldsymbol{v} - \tilde{\boldsymbol{v}}_h\big)_{\mathcal{T}_h} \\
& - (\boldsymbol{f} - \boldsymbol{f}_h, \boldsymbol{v} - \tilde{\boldsymbol{v}}_h)_{\mathcal{T}_h} + \big\langle (\nu\nabla_w \boldsymbol{u}_h - p_h \mathbf{I})\boldsymbol{n}, \boldsymbol{v} - \tilde{\boldsymbol{v}}_h \big\rangle_{\partial\mathcal{T}_h} \\
\lesssim& \Bigg[\nu^{\frac{1}{2}} \|\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h\|_0 + \nu^{-\frac{1}{2}} \Bigg(\sum_{T\in\mathcal{T}_h} h_T^2 \big\|\boldsymbol{f}_h + \nu\nabla_h\cdot(\nabla_w \boldsymbol{u}_h) - \nabla_h p_h\big\|_{0,T}^2 \Bigg)^{\frac{1}{2}} \\
& + \nu^{-\frac{1}{2}} \Bigg(\sum_{T\in\mathcal{T}_h} h_T^2 \big\|(\boldsymbol{f} - \boldsymbol{f}_h)\big\|_{0,T}^2 \Bigg)^{\frac{1}{2}} \\
& + \sum_{e\in\varepsilon_h^0} \nu^{-\frac{1}{2}} h_e^{\frac{1}{2}} \big\| [\![(-\nu\nabla_w \boldsymbol{u}_h + p_h \mathbf{I})\boldsymbol{n}]\!] \big\|_{0,e} \Bigg] \cdot \nu^{\frac{1}{2}} \|\nabla \boldsymbol{v}\|_0 \\
\lesssim& \big(\eta_h + \operatorname{osc}(\boldsymbol{f}, \mathcal{T}_h) + \operatorname{osc}(\boldsymbol{g}, \varepsilon_h^{\partial})\big)\, \nu^{\frac{1}{2}} \|\nabla \boldsymbol{v}\|_0 ,
\end{aligned}$$

such that

$$\nu^{-1}\|p - p_h\|_0^2 \lesssim \eta_h^2 + \operatorname{osc}^2(\boldsymbol{f}, \mathcal{T}_h) + \operatorname{osc}^2(\boldsymbol{g}, \varepsilon_h^{\partial}),$$

so with the estimates (3.29) and (3.30) we complete the proof. ☐

3.2. Efficiency

We use the standard bubble function technique [20] to derive an efficiency estimate for the error estimator η_h. To simplify the discussion, let us assume that $\boldsymbol{g}$ is a continuous piecewise polynomial of degree $k+1$ or less with respect to ε_h^{∂}.

Lemma 3.4. *Let* $(\boldsymbol{u}, p)$ *be the solution of the Stokes problem* (1.1) *and* $(\boldsymbol{u}_h, p_h)$ *the solution of the weak Galerkin scheme* (2.5)-(2.6). *Then for any* $T \in \mathcal{T}_h$ *and* $e \in \varepsilon_h^0$,

$$\begin{aligned}
& h_T \big\|\boldsymbol{f}_h + \nu\nabla_h\cdot(\nabla_w \boldsymbol{u}_h) - \nabla_h p_h\big\|_{0,T} \\
\lesssim& \nu\|\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h\|_{0,T} + \|p - p_h\|_{0,T} + h_T\|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,T} , \qquad (3.31) \\
& h_e^{\frac{1}{2}} \big\| [\![(-\nu\nabla_w \boldsymbol{u}_h + p_h \mathbf{I})\boldsymbol{n}]\!] \big\|_{0,e} \\
\lesssim& \nu\|\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h\|_{0,w_e} + \|p - p_h\|_{0,w_e} + h_T\|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,w_e} . \qquad (3.32)
\end{aligned}$$

Proof. We first prove estimate (3.31). For any $T \in \mathscr{T}_h$, let $\lambda_1, \lambda_2, \cdots, \lambda_{d+1}$ $(d = 2, 3)$ be the barycentric coordinates on T. Take $\boldsymbol{v}_T = (\boldsymbol{f}_h + \nu \nabla_h \cdot (\nabla_w \boldsymbol{u}_h) - \nabla_h p_h)\boldsymbol{\phi}_T$ where $\phi_T = (d+1)^{d+1}\lambda_1\lambda_2, \cdots, \lambda_{d+1}$ is a bubble function on T, from (1.1) we have

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}_T)_T - (p, \nabla\cdot \boldsymbol{v}_T)_T = (\boldsymbol{f}, \boldsymbol{v}_T)_T .$$

Consequently,

$$\begin{aligned}
&\left\|\boldsymbol{f}_h + \nu\nabla_h \cdot (\nabla_w \boldsymbol{u}_h) - \nabla_h p_h\right\|_{0,T}^2 \\
\lesssim &\big(\boldsymbol{f}_h + \nu\nabla_h \cdot (\nabla_w \boldsymbol{u}_h) - \nabla_h p_h, \boldsymbol{v}_T\big)_T \\
= &\nu(\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h, \nabla \boldsymbol{v}_T)_T - (p - p_h, \nabla\cdot \boldsymbol{v}_T)_T - (\boldsymbol{f} - \boldsymbol{f}_h, \boldsymbol{v}_T)_T \\
\lesssim &\big(\nu\|\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h\|_{0,T} + \|p - p_h\|_{0,T} + h_T\|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,T}\big) h_T^{-1}\|\boldsymbol{v}_T\|_{0,T} ,
\end{aligned}$$

which leads to the estimate (3.31). Similarly, to prove the estimate (3.32), for any $e \in \varepsilon_h^0$ let ϕ_e be the corresponding edge/face bubble function that vanishes on ∂w_e. Then taking $\boldsymbol{v}_e = [\![(-\nabla_w \boldsymbol{u}_h + p_h \mathbf{I})\boldsymbol{n}]\!]\phi_e$ we have

$$\begin{aligned}
&\left\|[\![(-\nu\nabla_w \boldsymbol{u}_h + p_h \mathbf{I})\boldsymbol{n}]\!]\right\|_{0,e}^2 \\
\lesssim &\big\langle [\![(-\nu\nabla_w \boldsymbol{u}_h + p_h \mathbf{I})\boldsymbol{n}]\!], \boldsymbol{v}_e\big\rangle_e \\
= &\sum_{T\in w_e} \Big\{\big(-\nu\nabla_h \cdot (\nabla_w \boldsymbol{u}_h) + \nabla_h p_h, \boldsymbol{v}_e\big)_T - \nu(\nabla_w \boldsymbol{u}_h, \nabla \boldsymbol{v}_e)_T + (p_h, \nabla\cdot \boldsymbol{v}_e)_T\Big\} \\
= &\sum_{T\in w_e} \big\{-(\boldsymbol{f}_h + \nu\nabla_h \cdot (\nabla_w \boldsymbol{u}_h) - \nabla_h p_h, \boldsymbol{v}_e)_T - (\boldsymbol{f} - \boldsymbol{f}_h, \boldsymbol{v}_e)_T \\
&\qquad + \nu(\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h, \nabla \boldsymbol{v}_e)_T - (p - p_h, \nabla\cdot \boldsymbol{v}_e)_T\big\} ,
\end{aligned}$$

which combined with the properties of the bubble function ϕ_e and the estimate (3.31) implies

$$\begin{aligned}
\left\|[\![(-\nu\nabla_w \boldsymbol{u}_h + p_h \mathbf{I})\boldsymbol{n}]\!]\right\|_{0,e} \lesssim &\sum_{T\in w_e} \Big\{h_T^{\frac{1}{2}}\left\|\boldsymbol{f}_h + \nabla_h \cdot (\nabla_w \boldsymbol{u}_h) - \nabla_h p_h\right\|_{0,T} + h_T^{\frac{1}{2}}\|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,T} \\
&\qquad + \nu h_T^{-\frac{1}{2}}\|\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h\|_{0,T} + h_T^{-\frac{1}{2}}\|p - p_h\|_{0,T}\Big\} \\
\lesssim &\nu h_T^{-\frac{1}{2}}\|\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h\|_{0,w_e} + h_T^{-\frac{1}{2}}\|p - p_h\|_{0,w_e} + h_T^{\frac{1}{2}}\|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,w_e} ,
\end{aligned}$$

and hence (3.32). □

Lemma 3.5. *Let* $(\boldsymbol{u}, p)$ *be the solution of the Stokes problem* (1.1) *and* $(\boldsymbol{u}_h = \{\boldsymbol{u}_0, \boldsymbol{u}_b\}, p_h)$ *the solution of the weak Galerkin scheme* (2.5)-(2.6). *Then for any* $e \in \varepsilon_h$*,*

$$h_e^{-\frac{1}{2}}\|\mathbf{J}_e(\boldsymbol{u}_0)\|_{0,e} \lesssim \sum_{T\in w_e} \Big(\|\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h\|_{0,T} + h_T^{-\frac{1}{2}}\|\boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b\|_{0,\partial T}\Big) ,$$

such that

$$\eta_{J,h}^2 \lesssim \nu\|\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h\|_0^2 + \eta_{b,h}^2 .$$

Proof. The second estimate follows from the first one directly, so it suffices to prove the first one. Recalling that $\varepsilon_h = \varepsilon_h^0 \cup \varepsilon_h^\partial$, from the property of the L^2−projection $\boldsymbol{Q}_b^k$ and the estimate (3.9), for any $e \in \varepsilon_h^0$ we have

$$
\begin{aligned}
h_e^{-\frac{1}{2}} \|\mathbf{J}_e(\boldsymbol{u}_0)\|_{0,e} &= h_e^{-\frac{1}{2}} \|[\![\boldsymbol{u}_0]\!]\|_e \\
&= h_e^{-\frac{1}{2}} \left\| [\![\boldsymbol{u}_0 - \boldsymbol{u} - \boldsymbol{Q}_b^k(\boldsymbol{u}_0 - \boldsymbol{u}) + \boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b]\!] \right\|_e \\
&\lesssim \sum_{T \in w_e} \left(h_T^{-\frac{1}{2}} \left\| \boldsymbol{u}_0 - \boldsymbol{u} - \boldsymbol{Q}_b^k(\boldsymbol{u}_0 - \boldsymbol{u}) \right\|_{0,\partial T} + h_T^{-\frac{1}{2}} \|\boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b\|_{0,\partial T} \right) \\
&\lesssim \sum_{T \in w_e} \left(\|\nabla_h \boldsymbol{u}_0 - \nabla \boldsymbol{u}\|_{0,T} + h_T^{-\frac{1}{2}} \|\boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b\|_{0,\partial T} \right) \\
&\lesssim \sum_{T \in w_e} \left(\|\nabla_h \boldsymbol{u}_0 - \nabla_w \boldsymbol{u}_h\|_{0,T} + \|\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h\|_{0,T} + h_T^{-\frac{1}{2}} \|\boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b\|_{0,\partial T} \right) \\
&\lesssim \sum_{T \in w_e} \left(\|\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h\|_{0,T} + h_T^{-\frac{1}{2}} \|\boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b\|_{0,\partial T} \right).
\end{aligned}
$$

Similarly, from $\boldsymbol{u}_b = Q_b^k \boldsymbol{g}$ on e for any $e \in \varepsilon_h^\partial$ we obtain

$$
h_e^{-\frac{1}{2}} \|\mathbf{J}_e(\boldsymbol{u}_0)\|_{0,e} = h_e^{-\frac{1}{2}} \|\boldsymbol{u}_0 - \boldsymbol{g}\|_e \lesssim \sum_{T \in w_e} \left(\|\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h\|_{0,T} + h_T^{-\frac{1}{2}} \|\boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b\|_{0,\partial T} \right),
$$

to complete the proof. ☐

We now define a bilinear form

$$
\begin{aligned}
\mathscr{A}(\mathbf{L}, \boldsymbol{u}, p, \widehat{\boldsymbol{u}}; \mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}) := \; & \nu^{-1}(\mathbf{L}, \mathbf{G})_{\mathscr{T}_h} + (\boldsymbol{u}, \nabla \cdot \mathbf{G})_{\mathscr{T}_h} - (\boldsymbol{v}, \nabla \cdot \mathbf{L})_{\mathscr{T}_h} + (\nabla p, \boldsymbol{v})_{\mathscr{T}_h} \\
& - (\nabla q, \boldsymbol{u})_{\mathscr{T}_h} + \left\langle \widehat{\boldsymbol{v}}, (\mathbf{L} - p\mathbf{I})\boldsymbol{n} \right\rangle_{\partial \mathscr{T}_h} - \left\langle \widehat{\boldsymbol{u}}, (\mathbf{G} - q\mathbf{I})\boldsymbol{n} \right\rangle_{\partial \mathscr{T}_h} \\
& + \nu \sum_{T \in \mathscr{T}_h} \left\langle h_T^{-1}(\boldsymbol{Q}_b^k \boldsymbol{u} - \widehat{\boldsymbol{u}}), \boldsymbol{Q}_b^k \boldsymbol{v} - \widehat{\boldsymbol{v}} \right\rangle_{\partial T}. \qquad (3.33)
\end{aligned}
$$

Lemma 3.6. *Let $(\boldsymbol{u}, p)$ be the solution of the Stokes problem* (1.1) *and $(\boldsymbol{u}_h = \{\boldsymbol{u}_0, \boldsymbol{u}_b\}, p_h)$ the solution of the weak Galerkin scheme* (2.5)-(2.6). *Then the error equation*

$$
\mathscr{A}(\nu \nabla \boldsymbol{u} - \nu \nabla_w \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_0, p - p_h, \boldsymbol{u} - \boldsymbol{u}_b; \mathbf{G}_k, \boldsymbol{v}_{k+1}, q_k, \widehat{\boldsymbol{v}}_k) = 0 \qquad (3.34)
$$

holds for all $\mathbf{G}_k \in [P_k(\mathscr{T}_h)]^{d \times d}$, $\boldsymbol{v}_{k+1} \in [P_{k+1}(\mathscr{T}_h)]^d$, $q_k \in P_k(\mathscr{T}_h) \cap L_0^2(\Omega)$, $\widehat{\boldsymbol{v}}_k|_e \in [P_k(e)]^d$ for $e \in \varepsilon_h^0$ and $\widehat{\boldsymbol{v}}_k|_e = 0$ for $e \in \varepsilon_h^\partial$. In addition,

$$
\begin{aligned}
& \mathscr{A}(\nu \nabla \boldsymbol{u} - \nu \nabla_w \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_0, p - p_h, \boldsymbol{u} - \boldsymbol{u}_b; \nu \nabla \boldsymbol{u} - \nu \nabla_w \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_0, p - p_h, \boldsymbol{u} - \boldsymbol{u}_b) \\
&= \eta_{b,h}^2 + \nu \|\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h\|_0^2 + \nu \sum_{T \in \mathscr{T}_h} h_T^{-1} \|\boldsymbol{Q}_b^k \boldsymbol{u} - \boldsymbol{u}\|_{\partial T}^2. \qquad (3.35)
\end{aligned}
$$

Proof. Eq. (3.34) follows from the Stokes problem (1.1), the WG scheme (2.5)-(2.6), the definitions (2.3)-(2.4) and integration by parts. Using the property of the L^2—projection operator $\boldsymbol{Q}_b^k$, we have

$$\begin{aligned}\left\|\boldsymbol{Q}_b^k(\boldsymbol{u}-\boldsymbol{u}_0)-(\boldsymbol{u}-\boldsymbol{u}_b)\right\|_{e\cap\partial T}^2 &= \left\|(\boldsymbol{Q}_b^k\boldsymbol{u}-\boldsymbol{u})-(\boldsymbol{Q}_b^k\boldsymbol{u}_0-\boldsymbol{u}_b)\right\|_{e\cap\partial T}^2\\ &= \|\boldsymbol{Q}_b^k\boldsymbol{u}-\boldsymbol{u}\|_{e\cap\partial T}^2+\|\boldsymbol{Q}_b^k\boldsymbol{u}_0-\boldsymbol{u}_b\|_{e\cap\partial T}^2,\end{aligned}$$

which combined with Eq. (3.34) yields Eq. (3.35). □

Lemma 3.7. *Let* $(\boldsymbol{u},p)$ *be the solution of the Stokes problem* (1.1) *and* $(\boldsymbol{u}_h=\{\boldsymbol{u}_0,\boldsymbol{u}_b\},p_h)$ *the solution of the weak Galerkin scheme* (2.5)-(2.6). *Then*

$$\eta_{b,h}^2 \lesssim e_h^2 + osc^2(\boldsymbol{f},\mathscr{T}_h). \tag{3.36}$$

Proof. We take

$$\mathbf{G}=\nu\nabla\boldsymbol{u}-\nu\mathbb{Q}^k(\nabla\boldsymbol{u}),\quad \boldsymbol{v}=\boldsymbol{u}-\boldsymbol{Q}_0^{k+1}\boldsymbol{u},\quad q=p-\mathscr{J}^kp,\quad \widehat{\boldsymbol{v}}=\boldsymbol{u}-\boldsymbol{Q}_b^k\boldsymbol{u}, \tag{3.37}$$

where $\mathbb{Q}^k,\boldsymbol{Q}_0^{k+1}$ and $\mathscr{J}^k$ are the L^2-projection operators from $[L^2(\Omega)]^{d\times d}$ onto $[P_k(\mathscr{T}_h)]^{d\times d}$, from $[L^2(\Omega)]^d$ onto $[P_{k+1}(\mathscr{T}_h)]^d$ and from $L^2(\Omega)$ onto $P_k(\mathscr{T}_h)$, respectively. Then applying Lemma 3.6 we obtain

$$\begin{aligned}&\mathscr{A}(\nu\nabla\boldsymbol{u}-\nu\nabla_w\boldsymbol{u}_h,\boldsymbol{u}-\boldsymbol{u}_0,p-p_h,\boldsymbol{u}-\boldsymbol{u}_b;\mathbf{G},\boldsymbol{v},q,\widehat{\boldsymbol{v}})\\ &=\eta_{b,h}^2+\nu\|\nabla\boldsymbol{u}-\nabla_w\boldsymbol{u}_h\|_0^2+\nu\sum_{T\in\mathscr{T}_h}h_T^{-1}\|\boldsymbol{Q}_b^k\boldsymbol{u}-\boldsymbol{u}\|_{\partial T}^2.\end{aligned} \tag{3.38}$$

On the other hand, using integration by parts and noting $-\nu\Delta\boldsymbol{u}+\nabla p=\boldsymbol{f}$ and $\nabla\cdot\boldsymbol{u}=0$, we have

$$\begin{aligned}&\mathscr{A}(\nu\nabla\boldsymbol{u}-\nu\nabla_w\boldsymbol{u}_h,\boldsymbol{u}-\boldsymbol{u}_0,p-p_h,\boldsymbol{u}-\boldsymbol{u}_b;\mathbf{G},\boldsymbol{v},q,\widehat{\boldsymbol{v}})\\ &=(\nabla\boldsymbol{u}-\nabla_w\boldsymbol{u}_h,\mathbf{G})_{\mathscr{T}_h}+(\boldsymbol{u}-\boldsymbol{u}_0,\nabla\cdot\mathbf{G})_{\mathscr{T}_h}-\langle\boldsymbol{u}-\boldsymbol{u}_b,\mathbf{G}\boldsymbol{n}\rangle_{\partial\mathscr{T}_h}\\ &\quad-\nu\big(\boldsymbol{v},\Delta\boldsymbol{u}-\nabla_h\cdot(\nabla_w\boldsymbol{u}_h)\big)_{\mathscr{T}_h}+(\nabla p-\nabla_hp_h,\boldsymbol{v})_{\mathscr{T}_h}\\ &\quad+\big\langle(\nu\nabla\boldsymbol{u}-\nu\nabla_w\boldsymbol{u}_h-(p-p_h)\mathbf{I})\boldsymbol{n},\widehat{\boldsymbol{v}}\big\rangle_{\partial\mathscr{T}_h}+\big\langle(\boldsymbol{u}-\boldsymbol{u}_b)\cdot\boldsymbol{n},q\big\rangle_{\partial\mathscr{T}_h}-(\boldsymbol{u}-\boldsymbol{u}_0,\nabla_hq)_{\mathscr{T}_h}\\ &\quad+\nu\sum_{T\in\mathscr{T}_h}h_T^{-1}\big\langle\boldsymbol{Q}_b^k(\boldsymbol{u}-\boldsymbol{u}_0)-\boldsymbol{u}+\boldsymbol{u}_b,\boldsymbol{Q}_b^k\boldsymbol{v}-\widehat{\boldsymbol{v}}\big\rangle_{\partial T}\\ &=(\nabla\boldsymbol{u}_0-\nabla_w\boldsymbol{u}_h,\mathbf{G})_{\mathscr{T}_h}-\langle\boldsymbol{u}_0-\boldsymbol{u}_b,\mathbf{G}\boldsymbol{n}\rangle_{\partial\mathscr{T}_h}+\big(\boldsymbol{f}+\nu\nabla_h\cdot(\nabla_w\boldsymbol{u}_h)-\nabla_hp_h,\boldsymbol{v}\big)_{\mathscr{T}_h}\\ &\quad+\big\langle(\nu\nabla\boldsymbol{u}-\nu\nabla_w\boldsymbol{u}_h-(p-p_h)\mathbf{I})\boldsymbol{n},\widehat{\boldsymbol{v}}\big\rangle_{\partial\mathscr{T}_h}-(\nabla_h\cdot\boldsymbol{u}_0,q)_{\mathscr{T}_h}+\big\langle(\boldsymbol{u}_0-\boldsymbol{u}_b)\cdot\boldsymbol{n},q\big\rangle_{\partial\mathscr{T}_h}\\ &\quad+\nu\sum_{T\in\mathscr{T}_h}h_T^{-1}\big\langle\boldsymbol{Q}_b^k(\boldsymbol{u}-\boldsymbol{u}_0)-\boldsymbol{u}+\boldsymbol{u}_b,\boldsymbol{Q}_b^k\boldsymbol{v}-\widehat{\boldsymbol{v}}\big\rangle_{\partial T}.\end{aligned} \tag{3.39}$$

From Eqs. (3.37) and the properties of the projection operators $\mathbb{Q}^k$, $\boldsymbol{Q}_0^{k+1}$, $\boldsymbol{Q}_b^k$ and $\mathscr{J}^k$, for any $T\in\mathscr{T}_h$ we have

$$
\begin{aligned}
&(\nabla \boldsymbol{u}_0-\nabla_w\boldsymbol{u}_h,\mathbf{G})_T=0\,, \quad \big(\boldsymbol{f}_h+\nu\nabla_h\cdot(\nabla_w\boldsymbol{u}_h)-\nabla_h p_h,\boldsymbol{v}\big)_T=0\,,\\
&\langle \boldsymbol{Q}_b^k\boldsymbol{u}_0-\boldsymbol{u}_b,\widehat{\boldsymbol{v}}\rangle_{\partial T}=0\,, \qquad (\nabla_h\cdot\boldsymbol{u}_0,q)_T=0\,,\\
&\big\langle(\nu\nabla\boldsymbol{u}-\nu\nabla_w\boldsymbol{u}_h-(p-p_h)\mathbf{I})\boldsymbol{n},\widehat{\boldsymbol{v}}\big\rangle_{\partial\mathscr{T}_h}=\big\langle(\nu\nabla\boldsymbol{u}-\nu\nabla_w\boldsymbol{u}_h-(p-p_h)\mathbf{I})\boldsymbol{n},\widehat{\boldsymbol{v}}\big\rangle_{\varepsilon_h^\partial}\,,\\
&\langle \boldsymbol{Q}_b^k\boldsymbol{u}-\boldsymbol{Q}_b^k\boldsymbol{u}_0-\boldsymbol{u}+\boldsymbol{u}_b,\boldsymbol{Q}_b^k\boldsymbol{v}-\widehat{\boldsymbol{v}}\rangle_{\partial T}=\|\boldsymbol{Q}_b^k\boldsymbol{u}-\boldsymbol{u}\|_{\partial T}^2-\langle\boldsymbol{Q}_b^k\boldsymbol{u}_0-\boldsymbol{u}_b,\boldsymbol{v}\rangle_{\partial T}\,,
\end{aligned}
$$

which combined with Eqs. (3.38) and (3.39) imply

$$
\begin{aligned}
&\eta_{b,h}^2+\nu\|\nabla\boldsymbol{u}-\nabla_w\boldsymbol{u}_h\|_0^2\\
=&(\boldsymbol{f}-\boldsymbol{f}_h,\boldsymbol{v})_{\mathscr{T}_h}-\nu\sum_{T\in\mathscr{T}_h}h_T^{-1}\langle\boldsymbol{Q}_b^k\boldsymbol{u}_0-\boldsymbol{u}_b,\boldsymbol{v}\rangle_{\partial T}-\big\langle\boldsymbol{u}_0-\boldsymbol{u}_b,(\mathbf{G}-q\mathbf{I})\boldsymbol{n}\big\rangle_{\partial\mathscr{T}_h}\\
&+\big\langle(\nu\nabla\boldsymbol{u}-\nu\nabla_w\boldsymbol{u}_h-(p-p_h)\mathbf{I})\boldsymbol{n},\widehat{\boldsymbol{v}}\big\rangle_{\varepsilon_h^\partial}\\
=:&\boldsymbol{R}_1+\boldsymbol{R}_2+\boldsymbol{R}_3\,.
\end{aligned}\tag{3.40}
$$

where we denote

$$
\begin{aligned}
\boldsymbol{R}_1&=(\boldsymbol{f}-\boldsymbol{f}_h,\boldsymbol{v})_{\mathscr{T}_h}\,,\\
\boldsymbol{R}_2&=-\nu\big\langle h_T^{-1}(\boldsymbol{Q}_b^k\boldsymbol{u}_0-\boldsymbol{u}_b),\boldsymbol{v}\big\rangle_{\partial\mathscr{T}_h}\,,\\
\boldsymbol{R}_3&=\big\langle\boldsymbol{u}_0-\boldsymbol{u}_b,(-\mathbf{G}+q\mathbf{I})\boldsymbol{n}\big\rangle_{\partial\mathscr{T}_h}+\big\langle(\nu\nabla\boldsymbol{u}-\nu\nabla_w\boldsymbol{u}_h-(p-p_h)\mathbf{I})\boldsymbol{n},\widehat{\boldsymbol{v}}\big\rangle_{\varepsilon_h^\partial}\,.
\end{aligned}
$$

From the properties of the projection $\boldsymbol{Q}_0^{k+1}$, the Cauchy inequality, the trace inequality and Young's inequality we therefore have

$$
\begin{aligned}
\boldsymbol{R}_1&=(\boldsymbol{f}-\boldsymbol{f}_h,\boldsymbol{v})_{\mathscr{T}_h}\\
&=\big(\boldsymbol{f}-\boldsymbol{f}_h,(\boldsymbol{u}-\boldsymbol{u}_0)-\boldsymbol{Q}_0^{k+1}(\boldsymbol{u}-\boldsymbol{u}_0)\big)_{\mathscr{T}_h}\\
&\lesssim \mathrm{osc}^2(\boldsymbol{f},\mathscr{T}_h)+\nu\|\nabla\boldsymbol{u}-\nabla_h\boldsymbol{u}_0\|_0^2\,,
\end{aligned}\tag{3.41}
$$

$$
\begin{aligned}
\boldsymbol{R}_2&=-\nu\sum_{T\in\mathscr{T}_h}h_T^{-1}\big\langle\boldsymbol{Q}_b^k\boldsymbol{u}_0-\boldsymbol{u}_b,\boldsymbol{u}-\boldsymbol{u}_0-\boldsymbol{Q}_0^{k+1}(\boldsymbol{u}-\boldsymbol{u}_0)\big\rangle_{\partial T}\\
&\lesssim\nu\sum_{T\in\mathscr{T}_h}h_T^{-\frac{1}{2}}\|\boldsymbol{Q}_b^k\boldsymbol{u}_0-\boldsymbol{u}_b\|_{\partial T}\|\nabla\boldsymbol{u}-\nabla_h\boldsymbol{u}_0\|_T\\
&\le\frac{1}{8}\eta_{b,h}^2+C_1\nu\|\nabla\boldsymbol{u}-\nabla_h\boldsymbol{u}_0\|_0^2\,,
\end{aligned}\tag{3.42}
$$

where and henceforth C_i $(i=1,2,\cdots)$ denote different positive constants independent of the mesh size h and the viscosity coefficient ν.

To deal with the term $\boldsymbol{R}_3$, we denote $\mathbf{G}_k:=\nu\mathbb{Q}^k(\nabla\boldsymbol{u})-\nu\nabla_w\boldsymbol{u}_h$, $q_k:=\mathscr{J}^kp-p_h$ and let $\boldsymbol{u}_h^*\in[P_{k+1}(\mathscr{T}_h)]^d\cap[H^1(\Omega)]^d$ be as in (3.25). Now with $\boldsymbol{g}$ a continuous piecewise polynomial of degree $k+1$ or less with respect to ε_h^∂, we have $\boldsymbol{u} = \boldsymbol{u}_h^* = \boldsymbol{g}$ on ε_h^∂, which combined with $\boldsymbol{u}_b = Q_b^k \boldsymbol{g} = Q_b^k \boldsymbol{u}$ on ε_h^∂ and $\langle \boldsymbol{u}_h^* - \boldsymbol{u}_b, (\nabla \boldsymbol{u} - p\mathbf{I})\boldsymbol{n}\rangle_{\partial\mathcal{T}_h \backslash \varepsilon_h^\partial} = 0$ yields

$$
\begin{aligned}
\boldsymbol{R}_3 &= \left\langle \boldsymbol{u}_b - \boldsymbol{u}_0, (\nu\nabla\boldsymbol{u} - \nu\mathbb{Q}^k(\nabla\boldsymbol{u}) - (p - \mathscr{I}^k p)\mathbf{I})\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} \\
&\quad + \left\langle \nu(\nabla\boldsymbol{u} - \nabla_w \boldsymbol{u}_h)\boldsymbol{n} - (p - p_h)\boldsymbol{n}, \boldsymbol{u} - \boldsymbol{Q}_b^k \boldsymbol{u}\right\rangle_{\varepsilon_h^\partial} \\
&= \left\langle \boldsymbol{u}_h^* - \boldsymbol{u}_0, (\nu\nabla\boldsymbol{u} - \nu\nabla_w\boldsymbol{u}_h - (p - p_h)\mathbf{I})\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} + \left\langle \boldsymbol{u}_b - \boldsymbol{u}_h^*, \nu(\nabla\boldsymbol{u} - \nabla_w\boldsymbol{u}_h)\boldsymbol{n} - (p - p_h)\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} \\
&\quad + \left\langle \boldsymbol{Q}_b^k\boldsymbol{u}_0 - \boldsymbol{u}_b, (\mathbf{G}_k - q_k \boldsymbol{I})\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} + \left\langle \boldsymbol{u}_h^* - \boldsymbol{u}_b, \nu(\nabla\boldsymbol{u} - \nabla_w\boldsymbol{u}_h)\boldsymbol{n} - (p - p_h)\boldsymbol{n}\right\rangle_{\varepsilon_h^\partial}
\end{aligned}
$$

so

$$
\begin{aligned}
\boldsymbol{R}_3 &= \left\langle \boldsymbol{u}_h^* - \boldsymbol{u}_0, \nu(\nabla\boldsymbol{u} - \nabla_w\boldsymbol{u}_h)\boldsymbol{n} - (p - p_h)\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} + \Big\langle \boldsymbol{u}_b - \boldsymbol{u}_h^*, \nu(\nabla\boldsymbol{u} - \nabla_w\boldsymbol{u}_h)\boldsymbol{n} \\
&\quad - (p - p_h)\boldsymbol{n}\Big\rangle_{\partial\mathcal{T}_h \backslash \varepsilon_h^\partial} + \left\langle \boldsymbol{Q}_b^k\boldsymbol{u}_0 - \boldsymbol{u}_b, (\mathbf{G}_k - q_k\boldsymbol{I})\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} \\
&= \left\langle \boldsymbol{u}_h^* - \boldsymbol{u}_0, \nu(\nabla\boldsymbol{u} - \nabla_w\boldsymbol{u}_h)\boldsymbol{n} - (p - p_h)\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} \\
&\quad - \sum_{e\in\varepsilon_h^0} \left\langle \boldsymbol{u}_h^* - \boldsymbol{u}_b, [\![(-\nu\nabla_w\boldsymbol{u}_h + p_h\mathbf{I})\boldsymbol{n}]\!]\right\rangle_e + \left\langle \boldsymbol{Q}_b^k\boldsymbol{u}_0 - \boldsymbol{u}_b, (\mathbf{G}_k - q_k\boldsymbol{I})\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} \\
&:= \boldsymbol{R}_{31} + \boldsymbol{R}_{32} + \boldsymbol{R}_{33}\,.
\end{aligned}
\tag{3.43}
$$

From the estimate (3.27) and Lemmas 3.4 and 3.5 we have

$$
\begin{aligned}
\boldsymbol{R}_{31} &= \left\langle \boldsymbol{u}_h^* - \boldsymbol{u}_0, \nu(\nabla\boldsymbol{u} - \nabla_w\boldsymbol{u}_h)\boldsymbol{n} - (p - p_h)\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} \\
&= \nu(\nabla\boldsymbol{u}_h^* - \nabla_h\boldsymbol{u}_0, \nabla\boldsymbol{u} - \nabla_w\boldsymbol{u}_h)_{\mathcal{T}_h} - (\nabla\cdot\boldsymbol{u}_h^* - \nabla_h\cdot\boldsymbol{u}_0, p - p_h)_{\mathcal{T}_h} \\
&\quad - (\boldsymbol{u}_h^* - \boldsymbol{u}_0, \boldsymbol{f}_h + \nu\nabla_h\cdot(\nabla_w\boldsymbol{u}_h) - \nabla_h p_h)_{\mathcal{T}_h} - (\boldsymbol{u}_h^* - \boldsymbol{u}_0, \boldsymbol{f} - \boldsymbol{f}_h)_{\mathcal{T}_h} \\
&\lesssim \Bigg(\nu\|\nabla\boldsymbol{u} - \nabla_w\boldsymbol{u}_h\|_0 + \|p - p_h\|_0 + \sum_{T\in\mathcal{T}_h} h_T^2 \left\|\boldsymbol{f}_h + \nu\nabla_h\cdot(\nabla_w\boldsymbol{u}_h) - \nabla_h p_h\right\|_{0,T}^2 \Bigg)^{\frac{1}{2}} \\
&\quad + \sum_{T\in\mathcal{T}_h} h_T^2 \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,T}^2)^{\frac{1}{2}} \Bigg) \Bigg(\sum_{e\in\varepsilon_h} h_e^{-1} \|\mathbf{J}_e(\boldsymbol{u}_0)\|_{0,e}^2 \Bigg)^{\frac{1}{2}} \\
&\le \frac{1}{8}\eta_{b,h}^2 + C_2\big(\nu\|\nabla\boldsymbol{u} - \nabla_w\boldsymbol{u}_h\|_0^2 + \nu^{-1}\|p - p_h\|_0^2 + \mathrm{osc}^2(\boldsymbol{f}, \mathcal{T}_h)\big),
\end{aligned}
\tag{3.44}
$$

$$
\begin{aligned}
\boldsymbol{R}_{32} &= -\sum_{e\in\varepsilon_h^0} \left\langle \boldsymbol{u}_h^* - \boldsymbol{u}_b, [\![(-\nu\nabla_w\boldsymbol{u}_h + p_h\mathbf{I})\boldsymbol{n}]\!]\right\rangle_e \\
&= \sum_{e\in\varepsilon_h^0} \left\langle \boldsymbol{u}_h^* - \boldsymbol{u}_0, [\![(-\nu\nabla_w\boldsymbol{u}_h + p_h\mathbf{I})\boldsymbol{n}]\!]\right\rangle_e + \sum_{e\in\varepsilon_h^0} \left\langle \boldsymbol{Q}_b^k u_0 - \boldsymbol{u}_b, [\![(-\nu\nabla_w\boldsymbol{u}_h + p_h\mathbf{I})\boldsymbol{n}]\!]\right\rangle_e \\
&\lesssim \Bigg(\sum_{e\in\varepsilon_h^0} h_e^{-1}\|\mathbf{J}_e(\boldsymbol{u}_0)\|_{0,e}^2 + \sum_{T\in\mathcal{T}_h} h_T^{-1} \|(\boldsymbol{Q}_b^k\boldsymbol{u}_0 - \boldsymbol{u}_b)\|_{0,\partial T}^2 \Bigg)^{\frac{1}{2}} \Bigg(\sum_{e\in\varepsilon_h^0} h_e^{-1} \|[\![(-\nu\nabla_w\boldsymbol{u}_h + p_h\mathbf{I})\boldsymbol{n}]\!]\|_{0,e}^2 \Bigg)^{\frac{1}{2}} \\
&\le \frac{1}{8}\eta_{b,h}^2 + C_3\left(\nu\|\nabla\boldsymbol{u} - \nabla_w\boldsymbol{u}_h\|_0^2 + \nu^{-1}\|p - p_h\|_0^2 + \mathrm{osc}^2(\boldsymbol{f}, \mathcal{T}_h)\right).
\end{aligned}
\tag{3.45}
$$

For the term $\boldsymbol{R}_{33}$, we readily have

$$\begin{aligned}\boldsymbol{R}_{33} &= \langle \boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b, (\mathbf{G}_k - q_k \mathbf{I})\boldsymbol{n}\rangle_{\partial \mathscr{T}_h} \\ &\lesssim \sum_{T\in\mathscr{T}_h} h_T^{-\frac{1}{2}} \|\boldsymbol{Q}_b^k \boldsymbol{u}_0 - \boldsymbol{u}_b\|_{0,\partial T} \left(\nu \|\mathbb{Q}^k(\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h)\|_{0,T} + \|\mathscr{J}^k(p-p_h)\|_{0,T}\right) \\ &\leq \frac{1}{8}\eta_{b,h}^2 + C_4\left(\nu\|\nabla \boldsymbol{u} - \nabla_w \boldsymbol{u}_h\|_{\mathscr{T}_h}^2 + \nu^{-1}\|p-p_h\|_{\mathscr{T}_h}^2\right), \end{aligned} \tag{3.46}$$

and the desired estimate (3.36) follows from (3.40)-(3.46). □

Using the Lemmas 3.5 and 3.7, we finally obtain the following efficiency result.

Theorem 3.2 (Lower bound)**.** *Let* $(\boldsymbol{u}, p)$ *be the solution of the Stokes problem* (1.1) *and* $(\boldsymbol{u}_h, p_h)$ *the solution of the weak Galerkin scheme* (2.5)-(2.6)*. Then*

$$\eta_h^2 \lesssim e_h^2 + osc^2(\boldsymbol{f}, \mathscr{T}_h)\,. \tag{3.47}$$

4. Numerical Results

We considered several numerical examples in two dimensions, in order to illustrate the reliability and efficiency of our residual-based *a posteriori* error estimator η_h established in Theorems 3.1-3.2 for the WG sheme (2.5)-(2.6) with $k = 0, 1$. In Examples 4.1-4.3, we set $\Omega = [0,1]\times[0,1]$ and adopted uniform mesh refinement of Ω (cf. Fig. 1), while for Example 4.4 we considered an L-shaped domain and used adaptive grids (cf. Fig. 2).

Example 4.1. This test is from Ref. [5], and we adopted the viscosity coefficient $\nu = 1$ and the source term $\boldsymbol{f}$ such that the analytical solution to the Stokes problem (1.1) is

$$\begin{aligned}&u_1(x,y) = -x^2(x-1)^2 y(y-1)(2y-1)\,, \quad u_2(x,y) = x(x-1)(2x-1)y^2(y-1)^2\,, \\ &p(x,y) = x^6 - y^6\,.\end{aligned}$$

Example 4.2. This test is from Ref. [16], and we set $\nu = 1$ and $\boldsymbol{f} = \boldsymbol{0}$. The analytical solution to the Stokes problem (1.1) is

$$\begin{aligned}&u_1(x,y) = -e^x(y\cos y + \sin y)\,, \quad u_2(x,y) = e^x y \sin y\,, \\ &p(x,y) = 2e^x \sin y - 2(e-1)(1-\cos 1)\,.\end{aligned}$$

Example 4.3. For three different choices of ν (viz. $\nu = 1, 1e-6$ and $1e-9$), we adopted $\boldsymbol{f}$ such that the analytical solution to (1.1) is

$$\begin{aligned}&u_1(x,y) = \sin 2\pi x \cos 2\pi y\,, \quad u_2(x,y) = -\cos 2\pi x \sin 2\pi y\,, \\ &p(x,y) = x^2y^2 - \frac{1}{9}\,.\end{aligned}$$

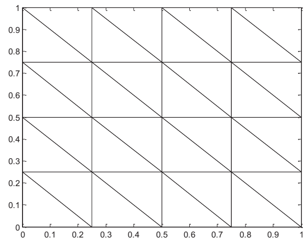
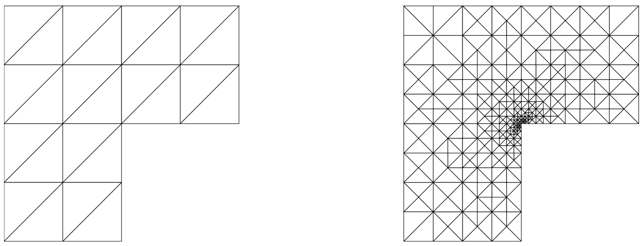


Figure 1: 4×4 grid of Ω in Examples 4.1-4.3.



(a) initial grid (b) adaptive grid ($l=16$)

Figure 2: Initial and adaptive grids of Ω in Example 4.4.

Example 4.4. This test is from Ref. [14]. We set $\Omega=(-1,1)^2\backslash[0,1)\times(-1,0]$ (cf. Fig. 2), $\boldsymbol{f}=0$, and $\nu=1$. The analytical solution to the Stokes problem (1.1) in polar coordinates (r,φ) is

$$
\begin{aligned}
u_1(r,\varphi)&=r^{\lambda}\big[(1+\lambda)\sin(\varphi)\Psi(\varphi)+\cos(\varphi)\Psi'(\varphi)\big],\\
u_2(r,\varphi)&=r^{\lambda}\big[\sin(\varphi)\Psi'(\varphi)-(1+\lambda)\cos(\varphi)\Psi(\varphi)\big],\\
p(r,\varphi)&=-r^{\lambda-1}\big[(1+\lambda)^2\Psi'(\varphi)+\Psi'''(\varphi)\big]/(1-\lambda),
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi(\varphi)=&\sin\big((1+\lambda)\varphi\big)\cos(\lambda\omega)/(1+\lambda)-\cos\big((1+\lambda)\varphi\big)\\
&-\sin\big((1-\lambda)\varphi\big)\cos(\lambda\omega)/(1-\lambda)+\cos\big((1-\lambda)\varphi\big),
\end{aligned}
$$

with the chosen parameters $\lambda=0.54448373678246$, $\omega=3\pi/2$.

In Tables 1-5 we show our numerical results for Examples 4.1-4.3 for the exact error e_h, the estimator η_h, the data oscillation terms $\mathrm{osc}(\boldsymbol{f},\mathscr{T}_h)$ and $\mathrm{osc}(\boldsymbol{g},\varepsilon_h^{\partial})$, the corresponding convergence rates, and the ratio $\gamma:=\eta_h/e_h$. In particular, Tables 3-5 show results for Example 4.3 for the three different viscosity coefficients ($\nu=1,1e-6$ and $1e-9$). In Example 4.1 we note that $\boldsymbol{g}=0$ means $\mathrm{osc}(\boldsymbol{g},\varepsilon_h^{\partial})=0$, and in Example 4.2 $\boldsymbol{f}=0$ implies $\mathrm{osc}(\boldsymbol{f},\mathscr{T}_h)=0$. Compared with the estimator η_h, the data oscillation terms $\mathrm{osc}(\boldsymbol{f},\mathscr{T}_h)$ and $\mathrm{osc}(\boldsymbol{g},\varepsilon_h^{\partial})$ are of higher order. We can also see that the ratio γ remains independent of the mesh size h and the viscosity coefficient ν as the mesh was refined, showing that

Table 1: Numerical results for Example 4.1.

k	Mesh	e_h	rate	η_h	rate	$\mathrm{osc}(\boldsymbol{f},\mathcal{T}_h)$	rate	γ
0	4×4	2.84E-01	-	3.90E-01	-	2.03E-02	-	1.37
	8×8	1.59E-01	0.83	1.94E-01	1.01	2.62E-03	2.96	1.22
	16×16	8.30E-02	0.94	9.67E-02	1.01	3.29E-04	2.99	1.17
	32×32	4.20E-02	0.98	4.82E-02	1.00	4.13E-05	3.00	1.15
	64×64	2.11E-02	1.00	2.40E-02	1.00	5.16E-06	3.00	1.14
1	4×4	4.28E-02	-	9.51E-02	-	1.59E-03	-	2.22
	8×8	1.16E-02	1.89	2.53E-02	1.91	1.01E-04	3.98	2.19
	16×16	2.97E-03	1.96	6.51E-03	1.96	6.33E-06	3.99	2.19
	32×32	7.51E-04	1.99	1.65E-03	1.98	3.96E-07	4.00	2.19
	64×64	1.89E-04	1.99	4.14E-04	1.99	2.48E-08	4.00	2.20

Table 2: Numerical results for Example 4.2.

k	Mesh	e_h	rate	η_h	rate	$\mathrm{osc}(\boldsymbol{g},\varepsilon_h^\partial)$	rate	γ
0	4×4	7.54E-01	-	4.80E-01	-	2.78E-01	-	0.64
	8×8	3.88E-01	0.96	2.66E-01	0.85	9.86E-02	1.50	0.69
	16×16	1.96E-01	0.99	1.39E-01	0.94	3.49E-02	1.50	0.71
	32×32	9.81E-02	1.00	7.05E-02	0.98	1.23E-02	1.50	0.72
	64×64	4.90E-02	1.00	3.55E-02	0.99	4.36E-03	1.50	0.72
1	4×4	3.72E-02	-	1.32E-02	-	1.30E-02	-	0.36
	8×8	9.51E-03	1.97	3.46E-03	1.94	2.31E-03	2.50	0.36
	16×16	2.41E-03	1.98	8.83E-04	1.97	4.08E-04	2.50	0.37
	32×32	6.05E-04	1.99	2.23E-04	1.99	7.21E-05	2.50	0.37
	64×64	1.52E-04	2.00	5.59E-05	1.99	1.28E-05	2.50	0.37

Table 3: Numerical results for Example 4.3 with $\nu=1$.

k	Mesh	e_h	rate	η_h	rate	$\mathrm{osc}(\boldsymbol{f},\mathcal{T}_h)$	rate	$\mathrm{osc}(\boldsymbol{g},\varepsilon_h^\partial)$	rate	γ
0	4×4	5.11E+00	-	8.09E+00	-	1.46E+00	-	1.93E+00	-	1.58
	8×8	2.69E+00	0.93	3.95E+00	1.03	1.93E-01	2.92	7.05E-01	1.46	1.47
	16×16	1.36E+00	0.98	1.95E+00	1.01	2.44E-02	2.98	2.51E-01	1.49	1.43
	32×32	6.84E-01	1.00	9.72E-01	1.01	3.07E-03	3.00	8.90E-02	1.50	1.42
	64×64	3.42E-01	1.00	4.85E-01	1.00	3.84E-04	3.00	3.15E-02	1.50	1.42
1	4×4	1.53E+00	-	3.26E+00	-	3.30E-04	-	3.94E-01	-	2.13
	8×8	4.08E-01	1.91	8.91E-01	1.87	2.19E-05	3.92	7.16E-02	2.46	2.18
	16×16	1.04E-01	1.98	2.29E-01	1.96	1.39E-06	3.98	1.27E-02	2.49	2.20
	32×32	2.60E-02	1.99	5.76E-02	1.99	8.71E-08	3.99	2.26E-03	2.50	2.21
	64×64	6.52E-03	2.00	1.44E-02	2.00	5.45E-09	4.00	3.99E-04	2.50	2.21

the proposed *a posteriori* estimator was reliable and efficient, and also robust with respect to ν. The analytical solution in Example 4.4 has a singularity at the origin. We used an adaptive WG ($k=0$) algorithm (**Algorithm** 4.1) to compute the approximation solution. In this algorithm, we took the marking parameter $\theta=0.5$ and stopping criterion $\mathbf{tol}=10^{-8}$, denoted by $\eta(\boldsymbol{u}_l,\boldsymbol{f},\boldsymbol{g},\mathcal{T}_l)$ the estimator η_h on the triangulation $\mathcal{T}_h=\mathcal{T}_l$, and refined

Table 4: Numerical results for Example 4.3 with $\nu = 10^{-6}$.

k	Mesh	e_h	rate	η_h	rate	osc($f,\mathcal{T}_h$)	rate	osc($g,\varepsilon_h^{\partial}$)	rate	γ
0	4×4	5.88E+01	-	1.04E+02	-	1.40E+00	-	1.93E-03	-	1.78
	8×8	3.10E+01	0.92	5.07E+01	1.04	1.76E-01	2.99	7.05E-04	1.46	1.63
	16×16	1.56E+01	0.99	2.50E+01	1.02	2.20E-02	3.00	2.51E-04	1.49	1.60
	32×32	7.79E+00	1.00	1.24E+01	1.01	2.75E-03	3.00	8.90E-05	1.50	1.59
	64×64	3.89E+00	1.00	6.18E+00	1.01	3.44E-04	3.00	3.15E-05	1.50	1.59
1	4×4	5.53E+00	-	1.34E+01	-	7.74E-02	-	3.94E-04	-	2.42
	8×8	1.43E+00	1.95	3.38E+00	1.99	4.84E-03	4.00	7.16E-05	2.46	2.37
	16×16	3.63E-01	1.98	8.49E-01	1.99	3.03E-04	4.00	1.27E-05	2.49	2.34
	32×32	9.16E-02	1.99	2.13E-01	2.00	1.89E-05	4.00	2.26E-06	2.50	2.32
	64×64	2.30E-02	1.99	5.32E-02	2.00	1.18E-06	4.00	3.99E-07	2.50	2.31

Table 5: Numerical results for Example 4.3 with $\nu = 10^{-9}$.

k	Mesh	e_h	rate	η_h	rate	osc($f,\mathcal{T}_h$)	rate	osc($g,\varepsilon_h^{\partial}$)	rate	γ
0	4×4	1.86E+03	-	3.30E+03	-	4.42E+01	-	6.11E-05	-	1.78
	8×8	9.81E+02	0.92	1.60E+03	1.04	5.56E+00	2.99	2.23E-05	1.46	1.63
	16×16	4.94E+02	0.99	7.91E+02	1.02	6.96E-01	3.00	7.94E-06	1.49	1.60
	32×32	2.46E+02	1.00	3.92E+02	1.01	8.71E-02	3.00	2.81E-06	1.50	1.59
	64×64	1.23E+02	1.00	1.95E+02	1.01	1.09E-03	3.00	9.95E-07	1.50	1.59
1	4×4	1.75E+02	-	4.24E+02	-	2.45E+00	-	1.25E-05	-	2.42
	8×8	4.52E+01	1.95	1.07E+02	1.99	1.53E-01	4.00	2.26E-06	2.46	2.37
	16×16	1.15E+01	1.98	2.68E+01	1.99	9.57E-03	4.00	4.03E-07	2.49	2.34
	32×32	2.90E+00	1.99	6.72E+00	2.00	5.98E-04	4.00	7.13E-08	2.50	2.32
	64×64	7.27E-01	1.99	1.68E+00	2.00	3.74E-05	4.00	1.26E-08	2.50	2.31

marked elements by the newest vertex bisection [15]. Fig. 2 shows the initial grid and an adaptive grid with $l = 16$, and Fig. 3 displays the decay history of the exact error e_l and the estimator $\eta_l = \eta(\boldsymbol{u}_l, \boldsymbol{f}, \boldsymbol{g}, \mathcal{T}_l)$.

In Fig. 2(b) we see that the refinement concentrates around the origin, so the error estimator was able to capture the solution singularity. Fig. 3 shows that the adaptive algorithm achieved almost the optimal order convergence for both the error and estimator — i.e. $e_l \lesssim N^{-1/2}$ and $\eta_l \lesssim N^{-1/2}$, where N denotes the number of degrees of freedom. We also note that $h = \mathrm{O}(N^{-1/2})$ for the quasi-uniform grids in two dimensions.

5. Conclusion

We have proposed a simple residual-type a posteriori error estimator for a weak Galerkin finite element discretization of the Stokes equations in two and three dimensional spaces. Both theoretical analysis and numerical experiments show that the estimator is reliable, efficient and robust with respect to the viscosity coefficient ν.

Algorithm 4.1 Adaptive algorithm

Input:

$\mathscr{T}_0$: initial triangulation; $\boldsymbol{f}$: source term; $\boldsymbol{g}$: boundary condition;

tol: stopping criteria; $\theta \in (0,1)$: marking parameter.

Output:

$\mathscr{T}_J$: a triangulation; $(\boldsymbol{u}_J, p_J)$: WG finite element approximation on T_J;

$\eta = 1; l = 0$;

while $\eta \geq$ **tol**

SOLVE the system (2.5)-(2.6) with $k = 0$ on $\mathscr{T}_l$ to get the approximation solution $(\boldsymbol{u}_l, p_l)$;

ESTIMATE the error by $\eta = \eta(\boldsymbol{u}_l, \boldsymbol{f}, \boldsymbol{g}, \mathscr{T}_l)$;

MARK a set $\mathscr{M}_l \subset \mathscr{T}_l$ with minimum number such that

$$\eta^2(\boldsymbol{u}_l, \boldsymbol{f}, \boldsymbol{g}, \mathscr{M}_l) \geq \theta \eta^2(\boldsymbol{u}_l, \boldsymbol{f}, \boldsymbol{g}, \mathscr{T}_l)$$

REFINE element in $\mathscr{M}_l$ and necessary elements to a conforming triangulation $\mathscr{T}_{l+1}$;

$l = l + 1$;

end

$(\boldsymbol{u}_J, p_J) = (\boldsymbol{u}_l, p_l), \mathscr{T}_J = \mathscr{T}_l$;

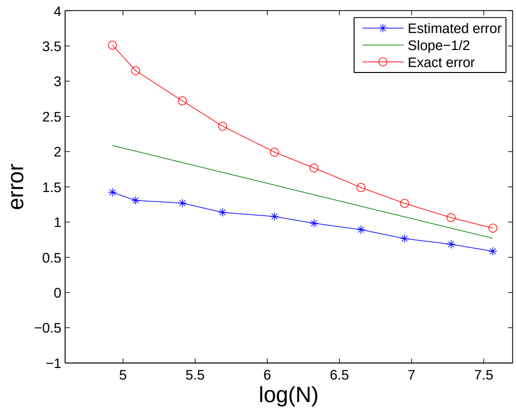


Figure 3: Decay history of the error and estimator.

Acknowledgments

This work was partially supported by the Major Research Plan of the National Natural Science Foundation of China (91430105).

References

[1] R. E. Bank and B. D. Welfert, *A posteriori error estimates for the Stokes equations: A comparison*, Comp. Meth. Appl. Mech. Engg. **82**, 323-340 (1990).

[2] R. E. Bank and B. D. Welfert, *A posteriori error estimates for the Stokes problem*, SIAM J. Num. An. **28**, 591-623 (1991).

[3] C. Carstensen, P. Causin and R. Sacco, *A posteriori dual-mixed adaptive finite element error control for Lamand Stokes equations*, Num. Mathematik **101**, 309-332 (2005).

[4] C. Carstensen and S. Funken, *A posteriori error control in low-order finite element discretisations of incompressible stationary flow problems*, Math. Comp. **70**, 1353-1381 (2001).

[5] G. Chen, M. Feng and X. Xie, *Robust globally divergence-free weak Galerkin methods for Stokes equations*, J. Comput. Math. **34**, 549-572 (2016).

[6] L. Chen, J. Wang and X. Ye, *A posteriori error estimates for weak Galerkin finite element methods for second order elliptic problems*, J. Scientific Comp. **59**, 496-511 (2014).

[7] S. Congreve, P. Houston, E. Suli and T.P. Wihler, *Discontinuous Galerkin finite element approximation of quasilinear elliptic boundary value problems II: Strongly monotone quasi-Newtonian flows*, IMA J. Num. Analysis **33**, 1386-1415 (2013).

[8] E. Dari, E. Durán and C. Padra, *Error estimators for nonconforming finite element approximations of the Stokes problem*, Math. Comp. **64**, 1017-1033 (1995).

[9] W. Dörfler and M. Ainsworth, *Reliable a posteriori error control for nonconforming finite element approximation of Stokes flow*, Math. Comp. **74** 1599-1619 (2005).

[10] M. Farhloul, S. Nicaise and L. Paque L, *A posteriori error estimation for the dual mixed finite element method of the Stokes problem*, C. R. Acad. Sci. Paris, Ser. I. **339**, 513-518 (2004).

[11] A. Hannukainen, R. Stenberg and M. VohralŠk, *A unified framework for a posteriori error estimation for the Stokes problem*, Num. Mathematik **122**, 725-769 (2012).

[12] D. Kay and D. Silvester, *A posteriori error estimation for stabilized mixed of the Stokes equations*, SIAM J. Scientific Comp. **21**, 1321-1336 (1999).

[13] R.J. Hosking and R.L. Dewar, *Fundamental Fluid Mechanics and Magnetohydrodynamics*, Springer (2016).

[14] P. Houston, D. Schotzau and T.P. Wihler, *hp-adaptive discontinuous Galerkin finite element methods for the Stokes proble*, European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS 2004), Eds. P. Neittaanmaki, P. Rossi, S. Korotov, E. Onate, J. Periaux and D. Knorzer, Jyvaskyla, 24-28 July 2004

[15] W.F. Mitchell, *A comparison of adaptive refinement techniques for elliptic problems*, ACM Trans. Math. Software (TOMS) **15**, 326-347 (1989).

[16] H. Paul, D. Schözau and T. P. Wihler, *Energy norm shape a posteriori error estimation for mixed discontinuous Galerkin approximations of the Stokes problem*, J. Scientific Comp. **22**, 347-370 (2005).

[17] Z. Shi and M. Wang, *Finite Element Methods*. Science Press (2013).

[18] R. Verfürth, *A posteriori error estimators for the Stokes equations*, J. Num. Mathematik, **55**, 309-325 (1989).

[19] R. Verfürth, *A posteriori error estimators for the Stokes equations II non-conforming discretizations*, J. Num. Mathematik **60**, 235-249 (1991).

[20] R. Verfürth, *A Review of A posteriori Error Estimation and Adaptive Mesh Refinement Techniques*, Wiley Teubner, Chichester and New York (1996).

[21] J. Wang and X. Ye, *A weak Galerkin finite element method for the Stokes equations*, Adv. Comp. Math. **42**, 1-20 (2015).

[22] R. Wang, X. Wang, Q. Zhai and R. Zhang, *A Weak Galerkin Finite Element Scheme for solving the stationary Stokes equations*, J. Comp. Appl. Math. **302**, 171-185 (2016).

[23] X. Zheng, G. Chen and X. Xie, *A divergence-free weak Galerkin method for quasi-Newtonian Stokes flows*, Science China Math. **60**, doi: 10.1007/s11425-016-0354-8 (2017).