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A Posteriori Error Estimator for a Weak Galerkin Finite Element Solution of the Stokes Problem

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Abstract. A robust residual-based *a posteriori* error estimator is proposed for a weak Galerkin finite element method for the Stokes problem in two and three dimensions. The estimator consists of two terms, where the first term characterises the difference between the L^2-projection of the velocity approximation on the element interfaces and the corresponding numerical trace, and the second is related to the jump of the velocity approximation between the adjacent elements. We show that the estimator is reliable and efficient through two estimates of global upper and global lower bounds, up to two data oscillation terms caused by the source term and the nonhomogeneous Dirichlet boundary condition. The estimator is also robust in the sense that the constant factors in the upper and lower bounds are independent of the viscosity coefficient. Numerical results are provided to verify the theoretical results.

AMS subject classifications: 65N15, 65N30, 76D07

Key words: The Stokes equations, weak Galerkin method, *a posteriori* error estimator.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded polygonal or polyhedral domain. We consider the following generalised Stokes problem: find the velocity $\boldsymbol{u}$ and the pressure p such that

$$\begin{aligned} & -\nu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \\ & \nabla\cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \\ & \boldsymbol{u} = \boldsymbol{g} \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

where $\nu > 0$ denotes the viscosity coefficient, Δ denotes the Laplacian differential operator, $\boldsymbol{f} \in [L^2(\Omega)]^d$ is the body force and $\boldsymbol{g}$ satisfies the compatibility condition $\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n} = 0$, with $\boldsymbol{n}$ the unit outward vector normal to the boundary $\partial\Omega$.

These equations describe steady viscous incompressible flow, and the development of reliable and efficient *a posteriori* error estimators for finite element discretisations of this problem has become an active research area in recent decades — cf. Refs. [1–4,7–11,13,14,

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16, 18, 19] and references therein. Specifically, two *a posteriori* error estimators have been produced for the mini-element based on the residual of the finite element solution and the solution of local problems [18], and related results can be found in Refs. [1,2]; *a posteriori* error estimators were analysed for non-conforming finite element approximations [8,9,19], discontinuous Galerkin methods [14, 16], and for dual mixed finite element methods [3, 10]; some unified framework for *a posteriori* error estimation for the Stokes problem based on H^1-conforming velocity reconstruction and H(div)-conforming locally conservative flux (stress) reconstruction has been provided [11]; and Refs. [4, 7] discuss *a posteriori* error analysis for quasi-Newtonian fluid flows.

A weak Galerkin (WG) finite element method for the Stokes equations (1.1) in the primary velocity-pressure formulation has been proposed [21]. The method uses a P_k/P_{k-1} ($k \geq 1$) discontinuous finite element combination for the velocity and pressure, with the velocity element being enhanced by polynomials of degree $k-1$ on the interface of the finite element partition. The usual gradient and divergence operators are implemented as distributions in properly-defined spaces.

Optimal-order error estimates were established for the corresponding numerical approximation in various norms. Refs. [5, 22] provide another two classes of WG methods for (1.1), and Ref. [23] presents a divergence-free WG method for quasi-Newtonian Stokes flows. Ref. [6] carried out the first *a posteriori* error analysis of WG methods for diffusion equations, where the residual type *a posteriori* error estimator is a combination of the standard conforming Galerkin and mixed finite elements.

Here we develop a residual type *a posteriori* error estimator for the WG method in Ref. [21] for the Stokes problem (1.1) in two and three dimensions. The *a posteriori* error estimator for the velocity error plus the pressure error consists of two terms. The first term characterises the difference between the L^2-projection of the velocity approximation on the element interfaces and the corresponding numerical trace, and the second term is related to the jump of the velocity approximation between the adjacent elements. We show that the estimator is reliable and efficient with two estimates of global upper and global lower bounds, up to two data oscillation terms caused by the source term and the nonhomogeneous Dirichlet boundary condition, and our *a posteriori* estimation is robust with respect to the viscosity coefficient. The main tool of our analysis is the Helmholtz decomposition for tensor fields.

In Section 2, we first provide some notation before proceeding to summarise the WG scheme [21]. We present our *a posteriori* error estimator in Section 3, and discuss its reliability and efficiency. Some relevant numerical results are produced in Section 4, and we make some final remarks in Section 5.

2. Weak Galerkin (WG) Scheme

2.1. Notation

For any bounded domain $D \subset \mathbb{R}^s$ ($s = d, d-1$), let $H^m(D)$ and $H_0^m(D)$ denote the m^{th} order Sobolev spaces on D, and $\|\cdot\|_{m,D}$, $|\cdot|_{m,D}$ the corresponding norm and semi-