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Efficient Preconditioner and Iterative Method for Large Complex Symmetric Linear Algebraic Systems

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Abstract. We discuss an efficient preconditioner and iterative numerical method to solve large complex linear algebraic systems of the form $(W + \mathrm{i}T)u = c$, where W and T are symmetric matrices, and at least one of them is nonsingular. When the real part W is dominantly stronger or weaker than the imaginary part T, we propose a block multiplicative (BM) preconditioner or its variant (VBM), respectively. The BM and VBM preconditioned iteration methods are shown to be parameter-free, in terms of eigenvalue distributions of the preconditioned matrix. Furthermore, when the relationship between W and T is obscure, we propose a new preconditioned BM method (PBM) to overcome this difficulty. Both convergent properties of these new iteration methods and spectral properties of the corresponding preconditioned matrices are discussed. The optimal value of iteration parameter for the PBM method is determined. Numerical experiments involving the Helmholtz equation and some other applications show the effectiveness and robustness of the proposed preconditioners and corresponding iterative methods.

AMS subject classifications: 15A06, 65F10, 65H10

Key words: Preconditioner, complex linear algebraic systems, Krylov subspace method, spectral properties, convergence.

1. Introduction

We consider large and sparse complex symmetric linear algebraic systems of the form

$$\mathbf{A}u = c\,, \quad \mathbf{A} \in \mathbb{C}^{n\times n} \;\text{ and }\; u, c \in \mathbb{C}^n \tag{1.1}$$

where $\mathbf{A} = W + \mathrm{i}T \in \mathbb{C}^{n\times n}$ is a complex nonsingular matrix with W and $T \in \mathbb{R}^{n\times n}$ both symmetric matrices, and at least one of them is nonsingular ($\mathrm{i} = \sqrt{-1}$). Such complex linear algebraic systems arise in many scientific and engineering applications — e.g., wave propagation [28], distributed control problems [22], structural dynamics [16], fast Fourier

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transform solution of time-dependent partial differential equations [13,14], molecular scattering and lattice quantum chromodynamics [20]. More detailed background and additional references can be found in Refs. [4,10,11]. Writing $u = x + \mathrm{i}y$ and $c = f + \mathrm{i}\,g$ with $x, y, f, g \in \mathbb{R}^n$, we can re-express the complex linear algebraic system (1.1) as the following equivalent real block two-by-two formulation [2,15]:

$$\mathscr{A}\tilde{z} \equiv \begin{pmatrix} W & -T \\ T & W \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv b\,. \tag{1.2}$$

Recently, many efficient iteration methods and preconditioning techniques have been developed for solving the complex linear algebraic system (1.1) or its equivalent real two-by-two block formulation (1.2) when W and T are symmetric and semi-positive definite and at least one of them is positive definite. For example, Bai *et al.* [5,7] proposed a modified Hermitian and skew-Hermitian splitting (MHSS) iteration method and a preconditioned MHSS iteration method for the linear algebraic system (1.1). They also proposed preconditioned MHSS (PMHSS) preconditioner for the linear algebraic system (1.2), which can be expressed as follows [8] :

$$P_{PMHSS} = \frac{1}{2\alpha}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}\begin{pmatrix} \alpha V + W & 0 \\ 0 & \alpha V + W \end{pmatrix}\begin{pmatrix} V^{-1} & 0 \\ 0 & V^{-1} \end{pmatrix}\begin{pmatrix} \alpha V + T & 0 \\ 0 & \alpha V + T \end{pmatrix}.$$

Salkuyeh *et al.* [27] presented a generalised SOR (GSOR) iterative method for (1.2), and to accelerate the convergence rate, Davod *et al.* [21] produced a preconditioned variant of this generalised SOR (PGSOR) iterative method. Their GSOR and PGSOR iterative methods can be associated with the respective preconditioners

$$P_{GSOR} = \frac{1}{\alpha}\begin{pmatrix} W & 0 \\ \alpha T & W \end{pmatrix} \quad \text{and} \quad P_{PGSOR} = \frac{1}{\alpha}\begin{pmatrix} \omega W + T & 0 \\ \alpha(\omega T - W) & \omega W + T \end{pmatrix}.$$

When the matrices W and T are symmetric indefinite and satisfy one of two assumptions $-W \preceq T \prec W$ or $T \preceq W \prec -T$, where $T \prec (\preceq) W$ means the matrix $W - T$ is symmetric positive (semi-) definite, a generalisation of the PMHSS iteration method (GMHSS) was presented by Xu [29]. Recently, Cao *et al.* [12] presented generalised PMHSS (GPMHSS) and generalised LPMHSS (GLPMHSS) iteration methods, and corresponding GPMHSS and GLPMHSS preconditioners for the case when $-T \prec W \preceq T$. Their theoretical results and numerical experiments showed that the GPMHSS iteration method and its preconditioner

$$P_{GPMHSS} = -\frac{1}{2\alpha\mathrm{i}}(\alpha V + W + T)(\alpha V - W + T)\,,$$

where α is a given positive constant and $V \in \mathbb{R}^{n\times n}$ is a prescribed symmetric positive definite matrix, proved more efficient than the proposed GMHSS or GLPMHSS methods with their preconditioners. Other iteration methods and preconditioning techniques to solve the complex symmetric linear algebraic system (1.1) can be found in Refs. [1,6,9,17–19,23–25,30].

Here we discuss effective iterative solution methods of the complex indefinite linear algebraic system (1.1) when T and W are symmetric and at least one of them is nonsingular.