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Computable Error Estimates for a Nonsymmetric Eigenvalue Problem

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Abstract. We provide some computable error estimates in solving a nonsymmetric eigenvalue problem by general conforming finite element methods on general meshes. Based on the complementary method, we first give computable error estimates for both the original eigenfunctions and the corresponding adjoint eigenfunctions, and then we introduce a generalised Rayleigh quotient to deduce a computable error estimate for the eigenvalue approximations. Some numerical examples are presented to illustrate our theoretical results.

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Key words: Nonsymmetric eigenvalue problem, computable error estimates, asymptotical exactness, finite element method, complementary method.

1. Introduction

The numerical solution of the nonsymmetric eigenvalue problem we discuss here is important in scientific and engineering computation — e.g. convection-diffusion problems in fluid mechanics and environmental applications [9, 12, 13]. Classical *a priori* error estimates only give the asymptotic convergence order in the standard Galerkin finite element method for the nonsymmetric eigenvalue problem [4], but *a posteriori* error estimates are of great importance for the adaptive finite element method in particular. More discussion of *a posteriori* error estimates can be found in Refs. [2, 5–7, 10, 12, 13, 15, 16] and other references therein.

Here we consider computable *a posteriori* error estimates for the eigenpair approximation of the nonsymmetric eigenvalue problem, solved by the conforming finite element method on general meshes. Our approach is based on the complementary energy

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method [11, 15–18]. Recently, the complementary energy method has been applied to derive the *a posteriori* error estimates for symmetric eigenvalue problems [21] and nonlinear eigenvalue problems [20]. It is well known that the nonsymmetric eigenvalue problem is always associated with an adjoint eigenvalue problem. Using the complementary energy method, we first derive asymptotic upper bounds for the error estimates of the original eigenfunction approximation and the adjoint eigenfunction approximation. Based on the *a posteriori* error estimates for the eigenfunction approximations and a generalised Rayleigh quotient, we then obtain asymptotic upper bounds for the error estimates of the eigenvalues by the conforming finite element method. This means we can provide a computable range of eigenvalues in the complex plane. Furthermore, the error estimates proposed here have both efficiency and reliability properties, which is necessary for the *a posteriori* error estimator.

The finite element method and corresponding error estimates for the nonsymmetric eigenvalue problem are given in Section 2. Asymptotic upper-bound computable error estimates of the original eigenfunction approximation and the adjoint eigenfunction approximation are proposed in Section 3. Based on the results in Section 3, in Section 4 we provide an upper bound for the error estimate of the eigenvalue approximations of the nonsymmetric eigenvalue problem. Some numerical examples are presented in Section 5 to illustrate the theoretical analysis, and our concluding remarks are made in Section 6.

2. Finite Element Method

We use the standard notation $W^{s,p}(\Omega)$ for Sobolev spaces, and $\|\cdot\|_{s,p,\Omega}$ and $|\cdot|_{s,p,\Omega}$ for their associated norms and seminorms, respectively — e.g. see Ref. [1]. For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega): \ v|_{\partial\Omega} = 0\}$, where $v|_{\partial\Omega} = 0$ is in the sense of trace, and $\|\cdot\|_{s,\Omega} = \|\cdot\|_{s,2,\Omega}$. Here we consider the complex Hilbert space $H_0^1(\Omega)$, and abbreviate $\|\cdot\|_{s,\Omega}$ as $\|\cdot\|_s$.

2.1. Nonsymmetric eigenvalue problem

For simplicity, we choose to consider the following nonsymmetric eigenvalue problem: Find $\lambda \in \mathscr{C}$ and u such that

$$\begin{cases} -\boldsymbol{\Delta} u + \mathbf{b}\cdot\boldsymbol{\nabla} u + u = \lambda u & \text{in } \Omega\,, \\ u = 0 \quad \text{on } \partial\Omega\,, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathscr{R}^d$ $(d = 2,3)$ is a bounded polygonal domain with boundary $\partial\Omega$, $\boldsymbol{\Delta}$ and $\boldsymbol{\nabla}$ respectively denote the Laplacian and gradient operator, and $\mathbf{b} = \mathbf{b}(\mathbf{x}) \in (W^{1,\infty}(\Omega))^d$ is a bounded real or complex vector function on Ω.

To address the finite element discretisation, we invoke the following variational form for the problem (2.1): Find $(\lambda, u) \in \mathscr{C} \times V$ such that

$$a(u,v) = \lambda(u,v)\,, \quad \forall\, v \in V\,, \tag{2.2}$$