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New Second-Order Schemes for Forward Backward Stochastic Differential Equations

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Abstract. The Feynman-Kac formulas are used to develop new second-order numerical schemes for the forward-backward stochastic differential equations (FBSDEs) of the first and second order. The methods are simple and allow an easy implementation. Numerous numerical tests for FBSDEs, fully nonlinear second-order parabolic partial differential equations and the Hamilton-Jacobi-Bellman equations show the stability and a high accuracy of the methods.

AMS subject classifications: 65C20, 65C30, 60H30, 60H35

Key words: Forward backward stochastic differential equations, Feynman-Kac formula, difference approximation, second-order scheme.

1. Introduction

Let $T > 0$ denote a deterministic terminal time and $(\Omega, \mathscr{F}, \mathbb{F}, P)$ be a filtered complete probability space with the natural filtration $\mathbb{F} = \{\mathscr{F}_t\}_{0\le t\le T}$ of an m-dimensional Brownian motion $W = (W_t)_{0\le t\le T}$. The decoupled forward-backward stochastic differential equation (FBSDE) on $(\Omega, \mathscr{F}, \mathbb{F}, P)$ has the form

$$
\begin{aligned}
X_t &= X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \\
Y_t &= \varphi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s,
\end{aligned}
\tag{1.1}
$$

where $t \in [0, T]$, $X_0 \in \mathscr{F}_0$ is an initial condition, $b : [0, T]\times\mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T]\times\mathbb{R}^d \to \mathbb{R}^{d\times m}$ are, respectively, drift and diffusion coefficients, $\varphi : \mathbb{R}^d \to \mathbb{R}^n$ and $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \to \mathbb{R}^n$ is a driver function. A triple (X_t, Y_t, Z_t) is called the L^2-adapted solution of the FBSDE (1.1) if it is $\mathscr{F}_t$-adapted, square integrable and satisfies the Eq. (1.1).

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We note that Pardoux and Peng [30] established the existence of a unique adapted solution for nonlinear backward stochastic differential equations (BSDEs). Using the results of Ref. [30], Peng [27] proposed a probabilistic interpretation of quasilinear parabolic partial differential equations (PDEs). Cheridito *et al.* [6] extended this interpretation to fully nonlinear parabolic PDEs by introducing a second-order forward-backward SDEs (2FBSDEs) and observing that the solutions of fully nonlinear parabolic PDE allow to solve the corresponding 2FBSDE. Since then FBSDEs are vigorously studied and applied in various fields, including mathematical finance [5,8,21,23], stochastic control [28], risk measure [15,28], mean-field BSDEs [2,3,32,33], and stochastic differential games [18].

Nevertheless, explicit closed-form solutions of FBSDEs can be rarely found, so that various numerical approaches to BSDEs, FBSDEs and 2FBSDEs have been proposed recently. In particular, Refs. [5, 7, 20, 23, 25, 44] use relationships between the solutions of BSDEs, FBSDEs, 2FBSDEs and the related PDEs [6, 21, 23, 27]. On the other hand, there are many approximation methods applied directly to BSDEs [4,16,36,39–41], FBSDEs [11–14,22,34,35,37,38,42,43,45] and 2FBSDEs [20,44]. For FBSDEs, most of the methods determine both Y_t and Z_t directly, which is a complicated and time consuming procedure, especially for Z_t in high dimensional situations.

In this paper, we propose new approximation methods for FBSDEs and 2FBSDEs. These methods are based, respectively, on the first-order [21,27] and the second-order Feynman-Kac formulas [20,44] and on the difference approximations of derivatives. They can be applied to nonlinear parabolic PDEs and Hamilton-Jacobi-Bellman (HJB) equations, arising in stochastic optimal control problems. The main features of the methods are:

- As soon as an approximation of Y_t is found, it is used in the approximation of Z_t.
- If the parameters of the methods are properly chosen, the accuracy of approximation of Y_t and Z_t can be of order 2 in time.
- The methods are simple and can be easily coded.
- The methods are applicable to fully nonlinear second-order parabolic PDEs.

Moreover, numerical experiments show that the methods are stable, efficient, and provide very accurate solutions of FBSDEs, 2FBSDEs, second-order nonlinear parabolic PDEs, and stochastic optimal control problems.

The paper is organised as follows. In Section 2 we recall the Feynman-Kac formulas and finite difference approximations. Section 3 deals with the discretisation of FBSDEs and 2FBSDEs. Novel fully discrete numerical methods for FBSDEs, 2FBSDEs and fully nonlinear parabolic PDEs are considered in Section 4. Here we also mention new backward stochastic differential equations and backward stochastic difference differential equations (BSDDEs). Numerical experiments reported in Section 5, demonstrate the efficiency and the accuracy of the methods, and our conclusions are in Section 6.