East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.240817.010318

Vol. **8**, No. 3, pp. 498-509
August 2018

Convergence of ADMM for Three-Block Separable Quadratic Programming Problems with Linear Constraints

Yifen Ke[1,2], Changfeng Ma[1] and Huai Zhang[2,3,*]

[1]*College of Mathematics and Informatics & FJKLMAA, Fujian Normal University, Fuzhou 350117, China.*
[2]*Key Laboratory of Computational Geodynamics, University of Chinese Academy of Sciences, Beijing 100049, China.*
[3]*Laboratory for Marine Mineral Resources, Qingdao National Laboratory for Marine Science and Technology, Qingdao 266237, China.*

Received 24 August 2017; Accepted (in revised version) 1 March 2018.

Abstract. The alternating direction method of multipliers is applied to three-block separable quadratic programming problems whose objective function is the sum of three functions without coupled variables. Necessary and sufficient conditions for the unique solvability of this problem are established. The convergence of the method is considered from the viewpoints of matrix computation and numerical optimisation.

AMS subject classifications: 90C30

Key words: Quadratic programming problem, alternating direction method of multipliers, splitting method.

1. Introduction

We consider the three-block separable quadratic programming problem with a linear constraint

$$\begin{aligned} &\min_{x,y,z} \phi_1(x)+\phi_2(y)+\phi_3(z), \\ &Ax+By+Cz=b, \end{aligned} \tag{1.1}$$

where $A\in\mathbb{R}^{p\times n}$, $B\in\mathbb{R}^{p\times m}$, $C\in\mathbb{R}^{p\times q}$, $b\in\mathbb{R}^p$ and the functions $\phi_1:\mathbb{R}^n\to\mathbb{R}$, $\phi_2:\mathbb{R}^m\to\mathbb{R}$ and $\phi_3:\mathbb{R}^q\to\mathbb{R}$ are defined by

$$\phi_1(x)=\frac{1}{2}x^{\mathrm{T}}Fx+f^{\mathrm{T}}x, \quad \phi_2(y)=\frac{1}{2}y^{\mathrm{T}}Gy+g^{\mathrm{T}}y, \quad \phi_3(z)=\frac{1}{2}z^{\mathrm{T}}Hz+h^{\mathrm{T}}z,$$

with symmetric positive semidefinite matrices $F\in\mathbb{R}^{n\times n}$, $G\in\mathbb{R}^{m\times m}$, $H\in\mathbb{R}^{q\times q}$ and $f\in\mathbb{R}^n$, $g\in\mathbb{R}^m$, $h\in\mathbb{R}^q$.

*Corresponding author. *Email addresses:* `hzhang@ucas.ac.cn` (H. Zhang)

The programming problem (1.1) occurs in various fields — e.g. in image alignment problem [16], principal component analysis models with noisy and incomplete data [9], Gaussian graphical model selection [5, 15], and quadratic discriminant analysis models [13]. The augmented Lagrangian function for the problem (1.1) is defined by

$$\mathscr{L}_A(x,y,z,\lambda) = \phi_1(x) + \phi_2(y) + \phi_3(z) - \langle Ax + By + Cz - b, \lambda\rangle + \frac{\alpha}{2}\|Ax + By + Cz - b\|_2^2,$$

where $\lambda \in \mathbb{R}^p$ is the Lagrange multiplier and $\alpha > 0$ a penalty parameter.

The augmented Lagrangian method (ALM) [10] is a benchmark for solving convex programming problems with linear constraints, and its iteration scheme for (1.1) is

$$\begin{aligned}(x_{k+1}, y_{k+1}, z_{k+1}) &:= \arg\min_{(x,y,z)} \mathscr{L}_A(x,y,z,\lambda_k),\\ \lambda_{k+1} &:= \lambda_k - \alpha(Ax_{k+1} + By_{k+1} + Cz_{k+1} - b).\end{aligned} \tag{1.2}$$

As is shown in [18], the iteration scheme (1.2) can be regarded as a dual ascent method over the dual variable λ, where the gradient of objective function for the dual of (1.1) is updated recursively by solving an (x,y,z)-minimisation problem over primal variable (x,y,z).

Let us now consider an extension of the alternating direction method of multipliers (ADMM) [7] for the problem (1.1) — viz.

$$\begin{aligned}x_{k+1} &:= \arg\min_{x\in\mathbb{R}^n} \mathscr{L}_A(x, y_k, z_k, \lambda_k),\\ y_{k+1} &:= \arg\min_{y\in\mathbb{R}^m} \mathscr{L}_A(x_{k+1}, y, z_k, \lambda_k),\\ z_{k+1} &:= \arg\min_{z\in\mathbb{R}^q} \mathscr{L}_A(x_{k+1}, y_{k+1}, z, \lambda_k),\\ \lambda_{k+1} &:= \lambda_k - \alpha(Ax_{k+1} + By_{k+1} + Cz_{k+1} - b).\end{aligned} \tag{1.3}$$

It can be interpreted as an alternating minimisation of the augmented Lagrangian function $\mathscr{L}_A(x,y,z,\lambda)$ successively with respect to x, y and z, followed by the update of the Lagrange multiplier λ. Various examples demonstrate the efficiency of the iteration procedure (1.3) — cf. Ref. [16, 19].

Nowadays, the ADMM is widely used in various applications. The convergence of the method is proved in the case where two blocks of variables are alternatively updated [10, 17]. Bai and Tao [3] established convergence for two-block separable quadratic programming problems with equality constraints. However, the direct generalisation of the ADMM to multi-block separable convex minimisation models with objective functions containing three or more functions without coupled variables may be divergent [6]. In particular, Cai *et al.* [4] presented a convex minimisation model with linear constraints and a separable objective function with three function blocks, such that the direct extension of the ADMM diverges. Nevertheless, they found sufficient conditions for the convergence of the method. In addition, Lin *et al.* [12] derived conditions of the global linear convergence of ADMM for convex programs, which minimise the sum of N convex functions with N-block variables connected by linear constraints. Besides, Lin *et al.* [11] proved that the