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Application of Reproducing Kernel Hilbert Spaces to a Minimization Problem with Prescribed Nodes

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Abstract. The theory of reproducing kernel Hilbert spaces is applied to a minimization problem with prescribed nodes. We re-prove and generalize some results previously obtained by Gunawan *et al*. [2,3], and also discuss the Hölder continuity of the solution to the problem.

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1. Introduction

Consider a Hilbert space H_α $(0 \le \alpha < \infty)$ defined by the set of functions f on $[0,1]^d$ of form

$$f(x_1,\dots,x_n) := \sum_{m_1,\dots,m_d\in\mathbb{N}} a_{m_1\cdots m_d} \sin(m_1\pi x_1)\cdots\sin(m_d\pi x_d)\,,$$

for which

$$\|f\|_{H_\alpha} := \frac{\pi^{2\alpha}}{2^d} \sum_{m_1,\dots,m_d\in\mathbb{N}} (m_1^2+\cdots+m_d^2)^\alpha |a_{m_1\cdots m_d}|^2 < \infty\,.$$

The above norm is induced from the inner product

$$\langle f,g\rangle_{H_\alpha} = \frac{\pi^{2\alpha}}{2^d} \sum_{m_1,\dots,m_d\in\mathbb{N}} (m_1^2+\cdots+m_d^2)^\alpha a_{m_1\cdots m_d} b_{m_1\cdots m_d}\,,$$

where $a_{m_1\cdots m_d}$ and $b_{m_1\cdots m_d}$ are the coefficients of f and g, respectively. We now proceed to solve the following minimization problem on $\mathbb{R}^d$:

$$\text{Minimize } \|f\|_{H_\alpha}$$

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subject to the prescribed nodes

$$f(\mathbf{p}_k) = c_k, \quad k = 1, \ldots, N \ ,$$

where $\mathbf{p}_k := (p_{k1}, \ldots, p_{kd}) \in (0,1)^d$ and $c_k \in \mathbb{R}$ are given. (Note that the points $\mathbf{p}_k$'s are inside the unit cube $[0,1]^d$.) The 1- and 2-dimensional cases have been studied by Gunawan *et al.* [2,3], who show inter alia that the value $\alpha > d/2$ is a necessary and sufficient condition for the solution to be continuous. As one might expect, the larger the value of α, the smoother the solution. Here we use the theory of reproducing kernel Hilbert spaces to study the problem in a more general setting.

Our first result is the following theorem.

Theorem 1.1. *Let $\alpha > d/2$. The solution to the minimization problem*

$$\text{Minimize } \|f\|_{H_\alpha}$$

subject to

$$f(p_1, \ldots, p_d) = 1 \ ,$$

is given by

$$F(x_1, \ldots, x_d) \ := \ A \sum_{m_1,\ldots,m_d \in \mathbb{N}} \frac{\sin(m_1 \pi p_1) \cdots \sin(m_d \pi p_d)}{(m_1^2 + \cdots + m_d^2)^\alpha} \sin(m_1 \pi x_1) \cdots \sin(m_d \pi x_d) \ ,$$

where $\quad A^{-1} \ := \sum_{m_1,\ldots,m_d \in \mathbb{N}} (\sin(m_1 \pi p_1) \cdots \sin(m_d \pi p_d))^2 / (m_1^2 + \cdots + m_d^2)^\alpha.$

The proof of this theorem is given in Section 2, where a more general result is also presented. In Section 3, we consider the Hölder continuity of the solution, by using the relationship between Besov and modulation spaces.

2. Main Results

Let E be a compact subspace of $\mathbb{R}^d$ containing at least N points, and $K : E \times E \to \mathbb{F}$ a positive definite kernel where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let H_K denote the corresponding reproducing kernel Hilbert space, which is defined as the completion of the pre-Hilbert space $H_K^0 := \operatorname{span}_{\mathbb{F}}\{K(\cdot, \mathbf{p}) : \mathbf{p} \in E\}$, equipped uniquely with the inner product $\langle \cdot, \cdot \rangle_{H_K^0}$ so that $\forall\, \mathbf{p}, \mathbf{q} \in E$

$$\langle K(\cdot, \mathbf{p}), K(\cdot, \mathbf{q}) \rangle_{H_K^0} = K(\mathbf{q}, \mathbf{p}) \, .$$

A well-known fact in the theory of reproducing kernel Hilbert spaces is that

$$f(\mathbf{p}) = \langle f, K(\cdot, \mathbf{p}) \rangle_{H_K}$$

for every $f \in H_K$ and $\mathbf{p} \in E$. Accordingly, we have the following proposition.

Proposition 2.1. *For every* $\mathbf{p} \in E$*,*

$$\{f \in H_K : f \perp K(\cdot, \mathbf{p})\} = \{f \in H_K : f(\mathbf{p}) = 0\}.$$

As a direct consequence, we obtain the following result, known as the representor theorem in learning theory (see e.g. [5]) and proved again here.

Proposition 2.2. *Let* $\mathbf{p} \in E$*, so that* $K(\mathbf{p}, \mathbf{p}) > 0$*. Then the minimization problem:*

$$\text{Minimize } \|f\|_{H_K}$$

subject to

$$f(\mathbf{p}) = 1$$

has a unique solution given by $F := \dfrac{K(\cdot, \mathbf{p})}{K(\mathbf{p}, \mathbf{p})}$.

Proof. Clearly $F(\mathbf{p}) = 1$. Now, if a function $g \in H_K$ satisfies $g(\mathbf{p}) = 1$, then

$$g - F \in \{f \in H_K : f(\mathbf{p}) = 0\} .$$

By Proposition 2.1, we have $g - F \perp K(\cdot, \mathbf{p})$, and accordingly $g - F \perp F$. It then follows that

$$\begin{aligned}
\|g\|_{H_K}^2 &= \|g - F\|_{H_K}^2 + 2\mathrm{Re}\langle g - F, F\rangle_{H_K} + \|F\|_{H_K}^2 \\
&= \|g - F\|_{H_K}^2 + \|F\|_{H_K}^2 \\
&\geq \|F\|_{H_K}^2,
\end{aligned}$$

and the equality is attained if and only if $g = F$. □

Theorem 1.1 can be seen to be a corollary of Proposition 2.2. Thus for $\mathbf{p} \in E := [0, 1]^d$, we may write

$$K_\alpha(\mathbf{x}, \mathbf{p}) := \frac{2^d}{\pi^{2\alpha}} \sum_{m_1, \ldots, m_d \in \mathbb{N}} \frac{\prod_{i=1}^d \sin(m_i \pi p_i)}{(m_1^2 + \cdots + m_d^2)^\alpha} \prod_{i=1}^d \sin(m_i \pi x_i) ,$$

where $\mathbf{x} = (x_1, \ldots, x_d) \in E$. Observe that $K(\mathbf{p}, \mathbf{p}) > 0$. Next, if $f(\mathbf{x}) = \prod_{i=1}^d \sin(M_i \pi x_i)$, then

$$\begin{aligned}
\langle f, K_\alpha(\cdot, \mathbf{p})\rangle &= \left\langle \prod_{i=1}^d \sin(M_i \pi \cdot_i), K(\cdot, \mathbf{p}) \right\rangle_{H_\alpha} \\
&= \frac{2^d}{\pi^{2\alpha}} \frac{\prod_{i=1}^d \sin(M_i \pi p_i)}{(M_1^2 + \cdots + M_d^2)^\alpha} \left\langle \prod_{i=1}^d \sin(M_i \pi \cdot_i), \prod_{i=1}^d \sin(M_i \pi \cdot_i) \right\rangle_{H_\alpha} \\
&= \prod_{i=1}^d \sin(M_i \pi p_i) \\
&= f(\mathbf{p}) .
\end{aligned}$$

It follows that $f(\mathbf{p}) = \langle f, K_\alpha(\cdot, \mathbf{p})\rangle$ for every $f \in H_\alpha$ and $p \in E$. Hence H_α is the reproducing kernel Hilbert space with kernel K_α, so the solution to the minimization problem

$$\text{Minimize } \|f\|_{H_\alpha}$$

subject to

$$f(\mathbf{p}) = 1$$

is $F(\mathbf{x}) := K(\mathbf{x}, \mathbf{p})/K(\mathbf{p}, \mathbf{p})$ — i.e.

$$F(x_1, \ldots, x_d) := A \sum_{m_1,\ldots,m_d \in \mathbb{N}} \frac{\sin(m_1\pi p_1)\cdots\sin(m_d\pi p_d)}{(m_1^2 + \cdots + m_d^2)^\alpha} \sin(m_1\pi x_1)\cdots\sin(m_d\pi x_d)\,,$$

where $$A^{-1} := \sum_{m_1,\ldots,m_d \in \mathbb{N}} (\sin(m_1\pi p_1)\cdots\sin(m_d\pi p_d))^2/(m_1^2 + \cdots + m_d^2)^\alpha.$$

The following proposition is more general than Proposition 2.2.

Proposition 2.3. *Consider a finite set* $\{c_k : k = 1, \ldots, N\} \subset \mathbb{F}$ *and* $P := \{\mathbf{p}_k : k = 1, \ldots, N\} \subset E$ *such that the matrix* $[K(\mathbf{p}_j, \mathbf{p}_k)]_{j,k=1}^N$ *is positive definite. Then the minimization problem*

$$\text{Minimize } \|f\|_{H_K}$$

subject to

$$f(\mathbf{p}_k) = c_k, \quad k = 1, \ldots, N$$

has a unique solution that lies in $\text{span}\{f_k : k = 1, \ldots, N\}$*, where* $f_k := K(\cdot, \mathbf{p}_k)$*.*

Proof. The system of linear equations

$$b_1 f_1(\mathbf{p}_j) + \cdots + b_N f_N(\mathbf{p}_j) = c_j, \quad j = 1, \ldots, N$$

is equivalent to

$$b_1 K(\mathbf{p}_j, \mathbf{p}_1) + \cdots + b_N K(\mathbf{p}_j, \mathbf{p}_N) = c_j, \quad j = 1, \ldots, N\,.$$

Since $[K(\mathbf{p}_j, \mathbf{p}_k)]_{j,k=1}^N$ is positive definite, the system has a unique solution, say $F := b_1 f_1 + \cdots + b_N f_N$. To prove that the norm is minimized, let $g \in H_K$ satisfy $g(\mathbf{p}_k) = c_k$ for $k = 1, \ldots, N$. Then, for each k, we have $(g - F)(\mathbf{p}_k) = 0$, so that $\langle g - F, f_k\rangle = 0$ or $g - F \perp f_k$. It follows that $g - F \perp F$, whence

$$\|g\|_{H_K}^2 = \|g - F\|_{H_K}^2 + \|F\|_{H_K}^2 \geq \|F\|_{H_K}^2\,,$$

so the equality is attained if and only if $g = F$. □

As a corollary, we obtain the following theorem.

Theorem 2.1. *Given $\mathbf{p}_k \in (0,1)^d$ and $c_k \in \mathbb{R}$ for $k = 1, \dots, N$, the minimization problem*

$$\text{Minimize } \|f\|_{H_\alpha}$$

subject to the prescribed nodes

$$f(\mathbf{p}_k) = c_k, \quad k = 1, \dots, N,$$

has the unique solution $F := b_1 f_1 + \cdots + b_N f_N$ where $f_k := K_\alpha(\cdot, \mathbf{p}_k)$, $k = 1, \dots, N$.

Proof. We need only ensure that the matrix $[K(\mathbf{p}_j, \mathbf{p}_k)]_{j,k=1}^N$ is positive definite, to deduce the existence and the uniqueness of the solution. Since the matrix is equal to the Gram matrix $[\langle K(\cdot, \mathbf{p}_k), K(\cdot, \mathbf{p}_j)\rangle_{H_\alpha}]_{j,k=1}^N$, it is sufficient to show that its determinant is nonzero. This is indeed so, because $\{\sin \pi x, \dots, \sin N\pi x\}$ forms a Chebyshev system (see [4]), and the product of such Chebyshev systems can always be used to interpolate data on any rectangular grid inside the cube $[0,1]^d$.

To illustrate this result, let us consider the 1-dimensional case. (For higher dimensional cases, we refer the reader to [1].) Our task reduces to verifying the linearly independence of the functions $K(\cdot, p_1), \dots, K(\cdot, p_N)$. Recall that for $k = 1, \dots, N$ we have

$$K(x, p_k) = \sum_{m=1}^{\infty} \frac{\sin m\pi p_k}{m^{2\alpha}} \sin m\pi x, \quad x \in [0,1].$$

Now the partial sums

$$K_N(x, p_k) = \sum_{m=1}^{N} \frac{\sin m\pi p_k}{m^{2\alpha}} \sin m\pi x, \quad x \in [0,1]$$

are linearly independent, since

$$\begin{vmatrix} \sin \pi p_1 & \frac{1}{2^{2\alpha}} \sin 2\pi p_1 & \cdots & \frac{1}{N^{2\alpha}} \sin N\pi p_1 \\ \sin \pi p_2 & \frac{1}{2^{2\alpha}} \sin 2\pi p_2 & \cdots & \frac{1}{N^{2\alpha}} \sin N\pi p_2 \\ \vdots & \vdots & \ddots & \vdots \\ \sin \pi p_N & \frac{1}{2^{2\alpha}} \sin 2\pi p_N & \cdots & \frac{1}{N^{2\alpha}} \sin N\pi p_N \end{vmatrix} \neq 0.$$

Hence the functions $K(\cdot, p_1), \dots, K(\cdot, p_N)$ must be linearly independent. □

3. Hölder continuity

We have seen that the solution to the minimization problem with several prescribed nodes is a linear combination of the minimizers with one prescribed node. Consequently, to study its Hölder continuity it suffices for us to investigate the Hölder continuity of the minimizer with one prescribed node, for which the formula is given in Theorem 1.1.

Theorem 3.1. *Assume $\alpha > d/2$ and let F be the solution to the minimization problem posed in* Theorem 1.1. *Put $\beta := \alpha - d/2$.*

1. *If $\beta \in \mathbb{N}$, then $F \in C^{\beta-1}(\mathbb{R}^d)$ with bounded partial derivatives up to order $\beta - 1$ and*

$$\sup_{x,y\in\mathbb{R}^d} \frac{|F^{(\beta-1)}(x) + F^{(\beta-1)}(y) - 2F^{(\beta-1)}((x+y)/2)|}{|x-y|} < \infty\,.$$

2. *If $\beta \in (0,\infty) \setminus \mathbb{N}$, then $F \in C^{[\beta]}(\mathbb{R}^d)$ and*

$$\sup_{x,y\in\mathbb{R}^d} \frac{|F^{([\beta])}(x) + F^{([\beta])}(y) - 2F^{([\beta])}((x+y)/2)|}{|x-y|^{\beta-[\beta]}} < \infty\,.$$

Theorem 3.1 depends upon the following lemmas, where $C^{\infty}_{\mathrm{comp}}(\mathbb{R}^d)$ denotes the set of all compactly supported smooth functions and $B(r)$ the open ball given by $\{x \in \mathbb{R}^d \,:\, |x| < r\}$ for $r > 0$.

Lemma 3.1 ([7, Theorem 4]). *Assume $\beta > 0$ and let $\psi \in C^{\infty}_{\mathrm{comp}}(\mathbb{R}^d)$ be such that*

$$\chi_{B(1)} \le \psi \le \chi_{B(2)}\,.$$

Define $\varphi_j(\xi) = \psi(2^{-j}\xi) - \psi(2^{-j+1}\xi)$ for $\xi \in \mathbb{R}^d$ and $j \in \mathbb{N}$.

1. *Let $G \in L^{\infty}(\mathbb{R}^d)$ and $\beta \in \mathbb{N}$. Then $G \in C^{\beta-1}(\mathbb{R}^d)$ with bounded partial derivatives up to order $\beta - 1$ and*

$$\sup_{x,y\in\mathbb{R}^d} \frac{|G^{(\beta-1)}(x) + G^{(\beta-1)}(y) - 2G^{(\beta-1)}((x+y)/2)|}{|x-y|} < \infty$$

if and only if

$$\|G\|_{B^{\beta}_{\infty,\infty}} = \|\mathscr{F}^{-1}[\psi \cdot \mathscr{F}G]\|_{\infty} + \sum_{j=1}^{\infty} 2^{j\beta} \|\mathscr{F}^{-1}[\varphi_j \cdot \mathscr{F}G]\|_{\infty} < \infty\,.$$

2. *If $\beta \in (0,\infty) \setminus \mathbb{N}$, then $G \in C^{[\beta]}(\mathbb{R}^d)$ with bounded partial derivatives up to order $[\beta]$ and*

$$\sup_{x,y\in\mathbb{R}^d} \frac{|G^{([\beta])}(x) + G^{([\beta])}(y) - 2G^{([\beta])}((x+y)/2)|}{|x-y|^{\beta-[\beta]}} < \infty$$

if and only if

$$\|G\|_{B^{\beta}_{\infty,\infty}} = \|\mathscr{F}^{-1}[\psi \cdot \mathscr{F}G]\|_{\infty} + \sum_{j=1}^{\infty} 2^{j\beta} \|\mathscr{F}^{-1}[\varphi_j \cdot \mathscr{F}G]\|_{\infty} < \infty\,.$$

The norm $\|\cdot\|_{B^{\beta}_{\infty,\infty}}$ in the above lemma is called the *Besov norm*.

Lemma 3.2 ([6]). *In the same notation as in Lemma* 3.1 *and assuming that $\alpha > d/2$, put $\beta := \alpha - d/2$. Define*

$$\tau_m = \psi(\cdot - m) \quad (m \in \mathbb{Z}^d),$$

and for $G \in L^{\infty}(\mathbb{R}^d)$ and $s \in \mathbb{R}$

$$\|G\|_{M^s_{\infty,2}} = \left(\sum_{m \in \mathbb{Z}^d} (1+|m|)^{2s} \|\mathscr{F}^{-1}[\tau_m \cdot \mathscr{F}G]\|_{\infty} \right)^{1/2}.$$

Then there exists a constant $C > 0$ such that

$$\|G\|_{B^{\beta}_{\infty,\infty}} \le C \|G\|_{M^{\alpha}_{\infty,2}}$$

for all $G \in L^{\infty}(\mathbb{R}^d)$.

With the above lemma, we observe that

$$\|F\|_{B^{\alpha-d/2}_{\infty,\infty}} \le C\|F\|_{M^{\alpha}_{\infty,2}} \le C\|F\|_{H_\alpha} < \infty,$$

and we obtain the proof of Theorem 3.1.

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