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Linearized Alternating Direction Method of Multipliers for Constrained Linear Least-Squares Problem

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Abstract. The alternating direction method of multipliers (ADMM) is applied to a constrained linear least-squares problem, where the objective function is a sum of two least-squares terms and there are box constraints. The original problem is decomposed into two easier least-squares subproblems at each iteration, and to speed up the inner iteration we linearize the relevant subproblem whenever it has no known closed-form solution. We prove the convergence of the resulting algorithm, and apply it to solve some image deblurring problems. Its efficiency is demonstrated, in comparison with Newton-type methods.

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1. Introduction

In this paper, we consider the constrained linear least-squares problem

$$\min_{\mathbf{l}\le\mathbf{x}\le\mathbf{u}}\left\{\frac{1}{2}\|A\mathbf{x}-\mathbf{c}\|^2+\frac{\lambda^2}{2}\|B\mathbf{x}-\mathbf{d}\|^2\right\}, \tag{1.1}$$

where $A,B\in\mathbb{R}^{m\times n}$ with $m\ge n$, $\mathbf{c},\mathbf{d}\in\mathbb{R}^m$, $\lambda\in\mathbb{R}$, $\mathbf{l}\in(\mathbb{R}\cup-\infty)^n$ and $\mathbf{u}\in(\mathbb{R}\cup+\infty)^n$ are given, and $\|\cdot\|$ denotes the 2-norm. The box constraints involved are to be interpreted entry-wise — i.e. $l_i\le x_i\le u_i$, $\forall i\in\{1,2,\cdots,n\}$.

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Clearly, the problem (1.1) can be written as $\min_{\mathbf{l}\leq\mathbf{x}\leq\mathbf{u}} \|C\mathbf{x}-\mathbf{e}\|^2/2$ for some $C \in \mathbb{R}^{2m\times n}$ and $\mathbf{e} \in \mathbb{R}^{2m}$, and there are standard solution procedures such as the Newton-type or interior point methods [9, 23, 29]. However, our emphasis here is on applications where A and B are two different types of operator, and the design of better algorithms to capture their properties — e.g. in image deblurring, where A is a blurring operator (integral operator) and B is a regularization operator (differential operator). When $\mathbf{d} = \mathbf{0}$ (1.1) becomes the Tikhonov regularization with $\mathbf{c}$ the observed image, λ^2 the regularization parameter, and $\mathbf{x}$ the image to be restored. The box constraints represent the dynamic range of the image — e.g. $l_i = 0$ and $u_i = 255$ for an 8-bit gray-scale image [37]. In our numerical experiments discussed in Section 4, by imposing the box constraints we find that the peak signal-to-noise ratio of the restored images can be increased by 0.2 to 2.2 decibles, so it pays to solve the constrained problem. In addition, (1.1) also serves as one of the two subproblems for the splitting algorithm in Ref. [22], which solves the deblurring problem with total-variational regularization where $B = I$ (the identity matrix) and $\mathbf{d}$ is an approximation of $\mathbf{x}$ [32]. Other applications of (1.1) include contact problems, control problems, and intensity-modulated radiotherapy problems [4, 10].

In this paper, we develop fast solvers for the problem (1.1) that exploit the properties of A and B. In the literature, algorithms for solving (1.1) are essentially Newton-type methods in the context of interior point methods — e.g. some that have been proposed to solve the nonnegative least-squares problem [9,29]. Interior point methods are particularly tailored to ill-posed problems arising in image reconstruction [5, 31]. By formulating the Karush-Kuhn-Tucker conditions as a system of nonlinear equations, a Newton-type method was proposed to solve the nonnegative least-squares problem [3]. A precondition technique was applied to deal with the resulting ill-conditioned inner linear system at each Newton iteration, when the iterate approaches a solution on the boundary of the feasible set. This work inspired the reduced Newton method presented in Ref. [24], to solve a subsystem of the inner linear system corresponding only to components of the iterate that are not close to the boundary. Since this subsystem is smaller and less ill-conditioned compared to the subsystem in Ref. [3], the reduced Newton method outperforms projection-type methods and some interior point Newton methods for image deblurring problems. A quite recent approach is the affine scaling method [7], which solves the Newton steps entry-wise by combining non-monotone line-search strategies and the cyclic version of the classical Barzilai-Borwein stepsize rule [1].

Here we apply the alternating direction method of multipliers (ADMM) originally proposed in Refs. [13, 15] to the problem (1.1), where we note the objective function is a sum of two least-squares terms linked together by a common variable $\mathbf{x}$. By introducing an auxiliary variable, we separate the two least-squares terms and apply the ADMM directly. The ADMM decomposes (1.1) into two easier subproblems, each of which is a least-squares problem with only one quadratic term in the objective function. However, if neither of the matrices A or B in (1.1) is the identity matrix so the respective ADMM subproblem does not have a closed-form solution, it is usually difficult to solve. In this paper, we apply a linearization such that a closed-form solution to the resulting linearized ADMM subproblem can be derived readily. For image deblurring problems, the cost per ADMM iteration