

Crank-Nicolson Quasi-Wavelet Based Numerical Method for Volterra Integro-Differential Equations on Unbounded Spatial Domains

Man Luo^{1,*}, Da Xu^{1,*} and Limei Li²

¹ Department of Mathematics, Hunan Normal University, 410081, Changsha, Hunan, China.

² Department of Mathematics, Hunan Institute of Science and Technology, 414000, Yueyang, Hunan, China.

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Abstract. The numerical solution of a parabolic Volterra integro-differential equation with a memory term on a one-dimensional unbounded spatial domain is considered. A quasi-wavelet based numerical method is proposed to handle the spatial discretisation, the Crank-Nicolson scheme is used for the time discretisation, and second-order quadrature to approximate the integral term. Some numerical examples are presented to illustrate the efficiency and accuracy of this approach.

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Key words: Parabolic Volterra integro-differential equation, unbounded spatial domain, quasi-wavelet, Crank-Nicolson method.

1. Introduction

Integro-differential equations are quite common in science and engineering — e.g. to describe porous viscoelastic behaviour with known fluctuations, or vibrations and dynamic populations. Various algorithms have been designed for the numerical solution of Volterra integro-differential equations — including finite element methods [1, 13–16], an orthogonal spline collection method [5] and finite difference methods. In particular, Xu discussed the numerical solution of a fractional diffusion equation by a finite difference scheme in time and a Legendre spectral scheme in space [10], Liu considered the numerical solution of the Rayleigh-Stokes problem involving a fractional derivative for a heated generalised second grade fluid [11], and Tang used the Crank-Nicolson scheme to approximate a partial integro-differential equation with a weakly singular kernel [12].

*Corresponding authors. Email addresses: mlwlx@163.com (M. Luo), daxu@hunnu.edu.cn (D. Xu), limeiliyy@163.com (L. Li)

In this article, we consider the numerical solution to the the following problem. Find $u(x, t)$ satisfying the Volterra integro-differential equation

$$\frac{\partial u(x, t)}{\partial t} + \int_0^t k(x, t-s)u(x, s)ds = \Delta u(x, t) + f(x, t), \quad x \in R, \quad t \in [0, T] \quad (1.1)$$

where $\Delta u = \partial^2 u / \partial x^2$, subject to the initial condition

$$u(x, 0) = u_0(x), \quad x \in R, \quad (1.2)$$

and the boundary condition

$$u \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \quad (1.3)$$

when the function $f(x, t)$ and the kernel function $k(x, t)$ are assumed to be sufficiently smooth.

When Eq. (1.1) applies on unbounded domains, numerical solutions have been obtained by many authors. One approach is the artificial boundary method to convert unbounded domain to bounded domains — e.g. Ma [19] used finite elements in space and the discontinuous Galerkin time-stepping method in time to solve the reduced problem, and the artificial boundary method for the numerical solution of parabolic PDEs on unbounded domains was considered in Refs. [20, 21]. An algebraic mapping has also been applied to the problem (1.1)-(1.3) on bounded domains, associated with the Legendre collocation method [22]. Here we use the Crank-Nicolson scheme is for the time discretisation, and the quasi-wavelet based numerical method for the spatial discretisation. The quasi-wavelet method is an effective way to approach the unbounded domain problem, since it is easy to implement and its distinctive local property produces accurate results. The localisation property allows us to analyse the local characteristics of functions involved [23], and the wavelet can be expressed as a superposition of its orthogonal scaling function. Thus the quasi-wavelet method is a very powerful tool for solving many kinds of partial integro-differential equations arising in real problems [17, 18, 24]. Interested readers may also refer to Refs. [28, 29, 35] for more detail on the quasi-wavelet numerical method.

We present the quasi-wavelet theory in Section 2. Subsection 3.1 presents the time discretisation for (1.1)-(1.3) via the Crank-Nicolson method, and the quasi-wavelet spatial discretisation and numerical algorithms are discussed in Section 3.2. Some numerical examples and results are given in Section 4, and concluding remarks in Section 5.

2. Quasi-Wavelet Based Numerical Method

Before giving a brief description of the quasi-wavelet based numerical method, let us first introduce the concept of singular convolution that often arise in science and engineering [25, 26]. A singular convolution is defined in the context of distribution theory as

$$F(t) = (T * g)(t) = \int_{-\infty}^{\infty} T(t-x)g(x)dx, \quad (2.1)$$

where T is a singular kernel. In this article, only singular kernels of delta type are required — i.e. such that

$$T(x) = \delta^{(q)}(x), \quad (q = 0, 1, 2, \dots), \quad (2.2)$$

where q denotes the q th-order derivative. However, since these singular kernels cannot be digitised directly in a computer, we construct a sequence of approximations δ_α to the distribution $\delta(x)$ such that

$$\lim_{\alpha \rightarrow \alpha_0} \delta_\alpha(x) \rightarrow \delta(x), \quad (2.3)$$

where α_0 is a generalised limit.[†]

Here we are interested in Shannon's delta kernel [30–32]

$$\delta_\alpha(x) = \frac{\sin(\alpha x)}{\pi x}, \quad (2.4)$$

one of the most important examples of the delta sequence kernel of Dirichlet. When $\alpha = \pi$, $\delta_\pi(x)$ is called Shannon's wavelet scaling function. The most important property of the Shannon delta kernel is that it provides an orthonormal basis in Hilbert space:

$$f(x) = \sum_{k \in \mathbb{Z}} \delta_\alpha(x - x_k) f(x_k), \quad \forall f \in B_\alpha^2, \quad (2.5)$$

where B_α^2 [20] is the Paley-Wiener reproducing kernel Hilbert space, and $\forall f \in B_\alpha^2$ indicates the function $f \in L^2$ in its Fourier representation vanishes outside the interval $[-\alpha, \alpha]$. The α is usually set equal to π/Δ , so

$$f(x) = \sum_{k \in \mathbb{Z}} \frac{\sin(\pi(x - x_k)/\Delta)}{\pi(x - x_k)/\Delta} f(x_k), \quad \forall f \in B_\pi^2, \quad (2.6)$$

where $\{x_k\}$ is an appropriate set of discrete points centred around the point x .

Unfortunately, Shannon's delta kernel decays slowly and leads to substantial errors. Indeed, strictly speaking the computation for Eq. (2.6) requires an infinite number of sampling points. In order to improve the localisation and asymptotic behaviour of Shannon's delta sequences kernel, a regularisation procedure has therefore been proposed, where the regularised orthogonal scaling function is defined as [27]

$$\delta_{\alpha,\sigma}(x) = \delta_\alpha(x) R_\sigma(x), \quad (\sigma > 0). \quad (2.7)$$

In particular, we choose $R_\sigma(x)$ to be a Gaussian regulariser such that

$$R_\sigma(x) = \exp(-x^2/2\sigma^2), \quad \sigma > 0, \quad (2.8)$$

$$\lim_{\sigma \rightarrow \infty} R_\sigma(x) = 1, \quad (2.9)$$

[†]Dirac first discussed the properties of the distribution $\delta(x)$ in his work on quantum mechanics (cf. his classic text "*Principles of Quantum Mechanics*" 4th Edition, Clarendon Press, Oxford 1958), so $\delta(x)$ is often called the Dirac delta function. Reference may also be made to Walter and Blum [33], for a discussion of the general orthogonal series analysis of the delta distribution and the numerical use of delta sequences as probability density estimations.

and

$$\int_{-\infty}^{\infty} \lim_{\sigma \rightarrow \infty} \delta_a(x) R_\sigma(x) dx = R_\sigma(0) = 1. \tag{2.10}$$

Here σ determines the width of the Gaussian envelope that is often varied according to the grid spacing Δ (i.e. $\sigma = r\Delta$), and the parameter r is chosen between 2.2 and 4.0 to generate good results. Combining Eqs. (2.6), (2.7) and (2.8), the regularised Shannon’s delta kernel on an arbitrary grid in its discrete form is given by

$$\delta_{\Delta,\sigma}(x - x_k) = \frac{\sin(\pi(x - x_k)/\Delta)}{\pi(x - x_k)/\Delta} \exp\left(-\frac{(x - x_k)^2}{2\sigma^2}\right). \tag{2.11}$$

We thus obtain a Schwartz class function to generate the quasi-wavelet to produce good numerical performance, and call this Gaussian regularised sampling function $\delta_{\Delta,\sigma}(x)$ a quasi-scaling function. In summary, an arbitrary continuous function f and its n th-order derivative are approximated by the discrete form

$$f^{(n)}(x) \approx \sum_{k=-W}^W \delta_{\Delta,\sigma}^{(n)}(x - x_k) f(x_k), \quad \delta_{\Delta,\sigma}^{(n)}(x - x_k) = \frac{d^n}{dx^n} \delta_{\Delta,\sigma}(x - x_k), \tag{2.12}$$

where $2W + 1$ is the computational bandwidth centred around x (usually smaller than the whole computational domain). We follow the choice of W, Δ, σ given in Ref. [30]. One problem is that $f(x_k)$ may locate outside of the computational domain $[a, b]$ near the computational boundary, so we must carefully choose the undefined function values — e.g. for Dirichlet boundary conditions $f(x_k) = f(a)$ or $f(b)$, for periodic boundary conditions by periodic mapping from the corresponding values inside the computational domain $[a, b]$, and for Neumann boundary conditions $f(x_k) = f'(a)$ or $f'(b)$. We adopt

$$\delta_{\Delta,\sigma}^{(1)}(x) = \begin{cases} -\frac{\cos(\pi x/\Delta)}{x} \exp\left(-\frac{x^2}{2\sigma^2}\right) - \frac{\sin(\pi x/\Delta)}{\pi x^2/\Delta} \exp\left(-\frac{x^2}{2\sigma^2}\right) \\ -\frac{\sin(\pi x/\Delta)}{\pi \sigma^2/\Delta} \exp\left(-\frac{x^2}{2\sigma^2}\right) & (x \neq 0) \\ 0 & (x = 0) \end{cases} \tag{2.13}$$

and

$$\delta_{\Delta,\sigma}^{(2)}(x) = \begin{cases} -\frac{\sin(\pi x/\Delta)}{x\Delta/\pi} \exp\left(-\frac{x^2}{2\sigma^2}\right) - 2\frac{\cos(\pi x/\Delta)}{x^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \\ -2\frac{\cos(\pi x/\Delta)}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) + 2\frac{\sin(\pi x/\Delta)}{\pi x^3/\Delta} \exp\left(-\frac{x^2}{2\sigma^2}\right) \\ +\frac{\sin(\pi x/\Delta)}{x\pi\sigma^2/\Delta} \exp\left(-\frac{x^2}{2\sigma^2}\right) + x\frac{\sin(\pi x/\Delta)}{\pi\sigma^4/\Delta} \exp\left(-\frac{x^2}{2\sigma^2}\right) & (x \neq 0) \\ -\frac{3 + \pi^2\sigma^2/\Delta^2}{3\sigma^2} & (x = 0). \end{cases} \tag{2.14}$$

3. Proposed Algorithms

We consider the numerical solution for the problem

$$\frac{\partial u(x, t)}{\partial t} + \int_0^t k_0(t-s)u(x, s)ds = \Delta u(x, t) + f(x, t), \quad x \in R, t \in [0, T], \quad (3.1)$$

$$u(x, 0) = u_0(x), \quad x \in R, \quad (3.2)$$

$$u \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \quad (3.3)$$

where $k_0(t-s) = e^{-(t-s)}$, $\Delta u = \partial^2 u / \partial x^2$ and the function $f(x, t)$ is assumed sufficiently smooth.

3.1. Discretisation in time: the Crank-Nicolson scheme

Let Δt be the time step, such that $t_k = k\Delta t$ for $k = 0, 1, \dots, N$. Let u^k denote the corresponding approximation to $u(x, t_k)$, and f^k the value of $f(x, t_k)$. The Crank-Nicolson discretisation scheme is

$$\frac{u^k - u^{k-1}}{\Delta t} = \frac{1}{2} \left(\Delta u^k + f^k - \int_0^{t_k} k_0(t_k-s)u(s)ds + \Delta u^{k-1} + f^{k-1} - \int_0^{t_{k-1}} k_0(t_k-s)u(s)ds \right), \quad (3.4)$$

where the integral may be approximated by the second-order quadrature rule — i.e.

$$\begin{aligned} \int_0^{t_k} k_0(t_k-s)u(x, s)ds &= \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} k_0(t_k-s)u(x, s)ds \\ &\approx \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} e^{-(t_k-s)} \left(u(x, t_i) \frac{s-t_{i+1}}{t_i-t_{i+1}} + u(x, t_{i+1}) \frac{s-t_i}{t_{i+1}-t_i} \right) ds \\ &= \frac{e^{-t_k}}{\Delta t} \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} e^s \left[u^{i+1}(s-t_i) - u^i(s-t_{i+1}) \right] ds \\ &= \frac{e^{-t_k}}{\Delta t} \sum_{i=0}^{k-1} \left[u^{i+1} \int_{t_i}^{t_{i+1}} e^s(s-t_i)ds - u^i \int_{t_i}^{t_{i+1}} e^s(s-t_{i+1})ds \right] \\ &= \frac{e^{-t_k}}{\Delta t} \sum_{i=0}^{k-1} \left[u^{i+1}(e^{t_{i+1}}\Delta t - e^{t_{i+1}} + e^{t_i}) + u^i(-e^{t_i}\Delta t + e^{t_{i+1}} - e^{t_i}) \right]. \end{aligned} \quad (3.5)$$

Invoking Eqs. (3.4) and (3.5) in Eq. (3.1) produces

$$\begin{aligned}
u^k - \frac{\Delta t}{2} \Delta u^k &= u^{k-1} + \frac{\Delta t}{2} \Delta u^{k-1} + \frac{\Delta t}{2} (f^k + f^{k-1}) \\
&\quad - \frac{\Delta t}{2} \frac{e^{-t_k}}{\Delta t} \sum_{i=0}^{k-1} \left[u^{i+1} (e^{t_{i+1}} \Delta t - e^{t_{i+1}} + e^{t_i}) + u^i (-e^{t_i} \Delta t + e^{t_{i+1}} - e^{t_i}) \right] \\
&\quad - \frac{\Delta t}{2} \frac{e^{-t_{k-1}}}{\Delta t} \sum_{i=0}^{k-2} \left[u^{i+1} (e^{t_{i+1}} \Delta t - e^{t_{i+1}} + e^{t_i}) + u^i (-e^{t_i} \Delta t + e^{t_{i+1}} - e^{t_i}) \right] \\
&= u^{k-1} + \frac{\Delta t}{2} \Delta u^{k-1} + \frac{\Delta t}{2} (f^k + f^{k-1}) \\
&\quad - \frac{1}{2} (e^{-t_k} + e^{-t_{k-1}}) \sum_{i=0}^{k-2} \left[u^{i+1} (e^{t_{i+1}} \Delta t - e^{t_{i+1}} + e^{t_i}) + u^i (-e^{t_i} \Delta t + e^{t_{i+1}} - e^{t_i}) \right] \\
&\quad - \frac{e^{-t_k}}{2} u^{k-1} (-e^{t_{k-1}} \Delta t + e^{t_k} - e^{t_{k-1}}) - \frac{e^{-t_k}}{2} u^k (e^{t_k} \Delta t - e^{t_k} + e^{t_{k-1}}),
\end{aligned}$$

hence we obtain the temporal semi-discrete form

$$\begin{aligned}
&u^k \left(\frac{1}{2} + \frac{1}{2} \Delta t + \frac{1}{2} e^{(t_{k-1}-t_k)} \right) - \frac{\Delta t}{2} \Delta u^k \\
&= u^{k-1} \left(\frac{1}{2} + \frac{1}{2} e^{(t_{k-1}-t_k)} \Delta t + \frac{1}{2} e^{(t_{k-1}-t_k)} \right) + \frac{\Delta t}{2} \Delta u^{k-1} + \frac{\Delta t}{2} (f^k + f^{k-1}) \\
&\quad - \frac{1}{2} (e^{-t_k} + e^{-t_{k-1}}) \sum_{i=0}^{k-2} \left[u^{i+1} (e^{t_{i+1}} \Delta t - e^{t_{i+1}} + e^{t_i}) + u^i (-e^{t_i} \Delta t + e^{t_{i+1}} - e^{t_i}) \right], \\
&\hspace{25em} k = 2, \dots, N. \tag{3.6}
\end{aligned}$$

In passing, we note that the first step of the Crank-Nicolson scheme is

$$\begin{aligned}
&u^1 \left(\frac{1}{2} + \frac{1}{2} \Delta t + \frac{1}{2} e^{-t_1} \right) - \frac{\Delta t}{2} \Delta u^1 \\
&= u^0 \left(\frac{1}{2} + \frac{1}{2} e^{(t_0-t_1)} \Delta t + \frac{1}{2} e^{(t_0-t_1)} \right) + \frac{\Delta t}{2} \Delta u^0 + \frac{\Delta t}{2} (f^1 + f^0). \tag{3.7}
\end{aligned}$$

3.2. Discretisation in space: the quasi-wavelet method

Given $x^0 \in R$ and $p \in (0, +\infty)$, we will now proceed to a detailed description of the spatial-temporal discretisation on an interval $[x^0 - p, x^0 + p]$. Consider a uniform grid

$$x_j = x^0 - p + j\Delta x, \quad \Delta x = 2p/M, \quad j = 0, 1, \dots, M$$

where M is a even number, so $x_j - x_{j+p} = -p\Delta x$. Let u_j^k denote an approximation to the value of $u(x_j, t_k)$. In the quasi-wavelet numerical method, only $2W$ grid points near the

point x are needed to approximate any function — e.g. the value of $u_x^{(n)}(x_j, t_k)$ can be approximated by

$$\begin{aligned} u_x^{(n)}(x_j, t_k) &\approx \sum_{m=j-W}^{j+W} \delta_{\Delta, \sigma}^{(n)}(x_j - x_m) u_m^k \\ &= \sum_{s=-W}^W \delta_{\Delta, \sigma}^{(n)}(-s\Delta x) u_{j+s}^k, \quad j = 0, 1, \dots, M. \end{aligned} \tag{3.8}$$

The relevant theorem on L_∞ error estimates for the quasi-wavelet scheme is as follows.

Theorem 3.1 (cf. [34]). *If the function $f(x) \in L_\infty \cap L_2(\Omega) \cap C^s(\Omega)$ is band-limited to B (i.e. $B < \alpha = \pi/\Delta$ where Δ is the grid spacing), $s \in \mathbb{Z}^+$, $\sigma = r\Delta > 0$ and $W \in \mathbb{N}$ is such that $W \geq sr/\sqrt{2}$, then*

$$\left\| f^{(s)} - \sum_{k=-W}^W \delta_{\Delta, \sigma}^{(s)}(x - x_k) f(x_k) \right\|_{L_\infty(\Omega)} \leq \beta \exp\left(-\frac{\gamma^2}{2r^2}\right).$$

Combining Eq. (3.6) and Eq. (3.8), the full discretisation of the problem (3.1)-(3.3) is

$$\begin{aligned} &u_j^k \left(\frac{1}{2} + \frac{1}{2}\Delta t + \frac{1}{2}e^{(t_{k-1}-t_k)} \right) - \frac{\Delta t}{2} \left(\sum_{s=-W}^W \delta_{\Delta, \sigma}^{(2)}(-s\Delta x) u_{j+s}^k \right) \\ &= u_j^{k-1} \left(\frac{1}{2} + \frac{1}{2}e^{(t_{k-1}-t_k)} \Delta t + \frac{1}{2}e^{(t_{k-1}-t_k)} \right) + \frac{\Delta t}{2} \sum_{s=-W}^W \delta_{\Delta, \sigma}^{(2)}(-s\Delta x) u_{j+s}^{k-1} + \frac{\Delta t}{2} (f_j^k + f_j^{k-1}) \\ &\quad - \frac{1}{2} (e^{-t_k} + e^{-t_{k-1}}) \sum_{i=0}^{k-2} \left[u_j^{i+1} (e^{t_{i+1}\Delta t} - e^{t_{i+1}} + e^{t_i}) + u_j^i (-e^{t_i\Delta t} + e^{t_{i+1}} - e^{t_i}) \right], \\ & \qquad \qquad \qquad j = 0, 1, \dots, M, \quad k = 2, \dots, N. \end{aligned} \tag{3.9}$$

The first step is

$$\begin{aligned} u_j^1 \left(\frac{1}{2} + \frac{1}{2}\Delta t + \frac{1}{2}e^{-t_1} \right) - \frac{\Delta t}{2} \sum_{s=-W}^W \delta_{\Delta, \sigma}^{(2)}(-s\Delta x) u_{j+s}^1 &= \frac{\Delta t}{2} (f_j^1 + f_j^0), \\ & \qquad \qquad \qquad j = 0, 1, \dots, M. \end{aligned} \tag{3.10}$$

Since some values of u^k are outside the spatial interval $[x^0 - p, x^0 + p]$ and the boundary condition is Dirichlet type, we set

$$u_j^k = \begin{cases} u_0^k & \text{if } j < 0, \\ u_M^k & \text{if } j > M. \end{cases}$$

4. Numerical Examples

Two numerical examples are considered, where we have set the parameters $T = 1$, $W = 35$ and $r = 3.2$.

Notes:

(1). Since the values at some points (such as u_0, u_M) are approximated, we only use the value of $u(x)$ in the interval $I = [u_W, u_{M-W}]$ when we calculate the relative ε_2 and ε_∞ errors. In particular, if $M < 2W$ we choose the interval $I = [u_{M/4}, u_{3M/4}]$ where M is a multiple of four.

(2). Because a very small time increment makes u feasible and the Shannon kernel decays exponentially with an increased number of spatial sampling points [34], the time step Δt must be chosen sufficiently small.

(3). We denote the computational solution and the exact solution by u^{comput} and u^{exact} , respectively. The ε_2 and ε_∞ errors are thus

$$\varepsilon_2 = \frac{\|u^{comput} - u^{exact}\|_{L_2(I)}}{\|u^{exact}\|_{L_2(I)}}, \quad \varepsilon_\infty = \frac{\|u^{comput} - u^{exact}\|_{L_\infty(I)}}{\|u^{exact}\|_{L_\infty(I)}}. \quad (4.1)$$

Example 4.1 (cf. [20]).

$$\frac{\partial u(x, t)}{\partial t} + \int_0^t k_0(t-s)u(x, s)ds = \Delta u(x, t) + f(x, t), \quad x \in R, t \in (0, T],$$

$$u|_{t=0} = 0, \quad x \in [0, +\infty),$$

$$u|_{x=0} = t, \quad t \in [0, T],$$

where $k_0(t) := e^{-t}$ and $f(x, t) := e^{-\beta x}(2 - e^{-t} - te^{-t} - \beta^2 t)$, $\beta > 0$. The exact solution is $u(x, t) = e^{-\beta x}t$.

To illustrate the resulting method, we choose $\beta = 5$ and show in Table 1 the resulting numerical values of ε_2 and ε_∞ in the interval $[0.5, 1.5]$ at the grid point $k = 100$, for two different grid sizes $\Delta t = 0.0001$ and $\Delta t = 0.00001$. Table 2 shows the errors in the interval $[35/20, 20 - 35/20]$. We make the following observations.

(1). When $\Delta t = 0.00001$ and $\varepsilon_\infty = 3.67941e - 014$, the numerical results have high accuracy.

(2). Comparing the results in Table 1 to the results in [20], we can see the computational solution in this paper is much better. For example, when $M = 32$ the ε^∞ error in Ref. [20] is $3.0881e-3$, whereas the ε^∞ error here is $1.44126e-004$. Furthermore, we find that the ε^2 and ε^∞ errors at $M = 64 - 256$ become progressively smaller.

(3). Due to the variability of x^0 and p , we can calculate over any arbitrary interval in R .

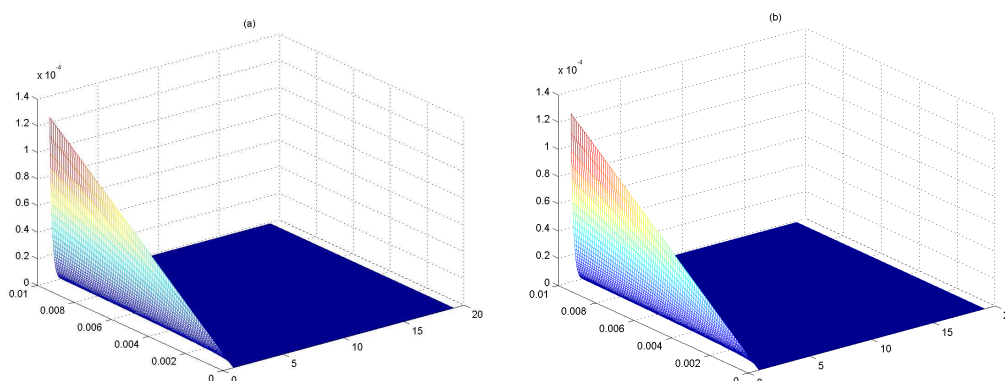
The surfaces of the computed and exact solutions are shown in Fig. 1(a) and in Fig. 1(b), respectively.

Table 1: ε_2 and ε_∞ errors of Example 4.1 at $x^0 = 1$, $p = 1$, $k = 100$.

M	$\Delta t = 0.0001$		$\Delta t = 0.00001$	
	ε_2	ε_∞	ε_2	ε_∞
4	1.72294e-001	2.12874e-001	1.80674e-002	2.24194e-002
8	8.34190e-002	9.43483e-002	1.26794e-002	1.55367e-002
16	1.36000e-002	1.34246e-002	5.38253e-003	6.60299e-003
32	1.37275e-004	1.44126e-004	1.19279e-004	8.65788e-005
64	2.67228e-006	4.66597e-006	2.50752e-007	2.07273e-007
128	1.11370e-006	1.86357e-006	2.38569e-011	4.06843e-011
256	4.81763e-007	8.04026e-007	2.37875e-011	4.06072e-011

Table 2: ε_2 and ε_∞ errors for Example 4.1 at $x^0 = 10$, $p = 10$, $k = 100$.

M	$\Delta t = 0.0001$		$\Delta t = 0.00001$	
	ε_2	ε_∞	ε_2	ε_∞
100	2.35827e-002	4.97121e-002	8.01143e-005	1.34096e-004
200	4.45337e-008	6.12512e-008	2.50891e-011	3.20006e-011
400	9.52259e-011	1.11937e-010	1.52735e-013	2.44454e-013
800	2.39996e-011	4.08519e-011	4.19038e-014	5.86386e-014
1600	2.38076e-011	4.06331e-011	2.24628e-014	3.67941e-014

Figure 1: (a) The numerical solution at $x^0 = 10$, $p = 10$, $k = 100$, $\Delta t = 0.0001$, $M = 800$ for Example 4.1; (b) exact solution $e^{-\beta x t}$.

Example 4.2 (cf. [22]). Let us now consider the problem (1.1)-(1.3) when $k(t) := 1$ and $x \in \mathbb{R}$, where the exact solution $u(x, t) = t e^{-x^2}$.

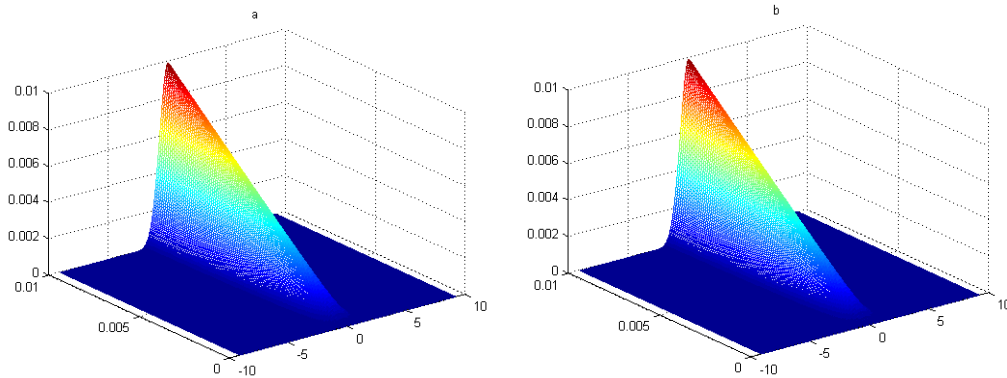
In Table 3 we show the relative ε^2 and ε^∞ errors at $x^0 = 0$, $p = 1$ (and domain $[-0.5, 0.5]$). Table 4 shows the error in the interval $[-10 + 35/20, 10 - 35/20]$ for $\Delta t = 0.0001$ and $\Delta t = 0.00001$, respectively. Fig. 2(a) shows the computed solution for $x_0 = 0$, $p = 10$, $M = 800$, $k = 100$ and $\Delta t = 0.0001$, and in Fig. 2(b) the exact solution is depicted. In comparison with the results in Ref. [22], the ε^∞ error here ($2.39340e-10$) is much smaller

Table 3: L^2 and L^∞ for Example 4.2 at $x^0 = 0$, $p = 1$, $k = 100$.

M	$\Delta t = 0.0001$		$\Delta t = 0.00001$	
	L^2	L^∞	L^2	L^∞
4	1.56820e-03	1.87821e-03	1.78261e-04	2.19369e-04
8	6.37213e-04	6.69075e-04	1.12543e-04	1.33797e-04
16	5.40021e-05	2.85448e-05	3.48775e-05	3.12997e-05
32	3.83397e-07	1.48567e-07	4.64002e-07	2.36856e-07
64	8.03188e-08	6.23181e-08	8.56738e-10	2.39340e-10

Table 4: L^2 and L^∞ of Example 4.2 at $x^0 = 0$, $p = 10$, $k = 100$.

M	$\Delta t = 0.0001$		$\Delta t = 0.00001$	
	L^2	L^∞	L^2	L^∞
100	1.37744e-13	1.51986e-13	1.36936e-14	1.47076e-14
200	5.16220e-15	5.48709e-15	2.89574e-15	3.10414e-15
400	5.23657e-15	5.48360e-15	6.84575e-15	6.68195e-15
800	7.19299e-15	8.74289e-15	3.58389e-15	4.20345e-15
1600	2.57970e-14	3.29171e-14	2.23716e-15	2.14314e-15

Figure 2: (a) Numerical solution at $x^0 = 0$, $p = 10$, $k = 100$, $\Delta t = 0.0001$, $M = 800$ and (b) exact solution $u(x, t) = te^{-x^2}$ for Example 4.2.

than the value $6.55e - 6$ in Ref. [22]. Furthermore, we can calculate any interval in R in constructing Table 3 and Table 4.

5. Conclusion

The quasi-wavelet based numerical method for solving parabolic Volterra-type integro-differential equations on one-dimensional unbounded spatial domains was discussed. We use the Crank-Nicolson method to discretise the time derivative, and then applied the quasi-wavelet method to discretise the spatial derivative. The efficiency and accuracy of

this approach was demonstrated in two examples. To the best of our knowledge, this is the first article where this class of problems has been tackled with the quasi-wavelet numerical method. Future work to solve the problem involving the integro-differential equation (1.1) on two-dimensional unbounded spatial domains via the Crank-Nicolson method is envisaged.

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