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Further Solutions of a Yang-Baxter-like Matrix Equation

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Abstract. The Yang-Baxter-like matrix equation $AXA = XAX$ is reconsidered, and an infinite number of solutions that commute with any given complex square matrix A are found. Our results here are based on the fact that the matrix A can be replaced with its Jordan canonical form. We also discuss the explicit structure of the solutions obtained.

AMS subject classifications: 15A18

Key words: Matrix equation, Jordan canonical form, projector.

1. Introduction

Let A be a complex $n \times n$ matrix in the quadratic matrix equation

$$AXA = XAX \tag{1.1}$$

for the unknown matrix $X \in \mathbf{C}^{n\times n}$, which we refer to as a *Yang-Baxter-like matrix equation* since its form is similar to the classic parameter-free Yang-Baxter equation originally introduced by Yang in 1967 [9] and then independently by Baxter five years later [2] in the field of statistical mechanics. The Yang-Baxter equation is also closely related to several mathematical areas such as braid groups and knot theory (e.g. see Refs. [8, 10] for more details and related topics), and some other applications have already appeared — e.g. see Refs. [1, 7].

Obviously, Eq. (1.1) has the two trivial solutions $X = 0$ and $X = A$, but its nonlinearity makes it difficult to find solutions in general. Recently, several classes of solutions of Eq. (1.1) have been obtained for some special cases of the given matrix A. When A is a nonsingular quasi-stochastic matrix such that A^{-1} is a stochastic matrix, the Brouwer fixed

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point theorem was used to prove that a solution exists and some numerical solutions were obtained via direct iteration [5]. When A is a projector or idempotent matrix (i.e. such that $A^2 = A$), all of the solutions of Eq. (1.1) have been found [3]. With the help of the spectral projection theorem in the analytic theory of matrices, a general spectral solution result of Eq. (1.1) was obtained, without any hypothesis about the given matrix A [6]. In particular, generalised eigenspaces and the concept of the index of an eigenvalue were used to explore the analytic properties of the given matrix A.

In this article, we continue our investigation of Eq. (1.1) to find further nontrivial solutions for a given matrix A. Until now, only finitely many solutions have been obtained, except when A is a projector. We recall that the centraliser of A, consisting of all of the solutions of the linear matrix equation $AX = XA$, is an n-dimensional subspace of $C^{n\times n}$ — cf. Theorem 5.16 of Ref. [4]. Similarly, except for the trivial case of $n = 1$ the general Yang-Baxter-like matrix equation has infinitely many solutions represented by a system of n^2 quadratic equations with n^2 unknowns, which constitute a sub-manifold of $\mathbf{C}^{n\times n}$. For instance, directly solving the 2×2 form of Eq. (1.1) with a simple Jordan block

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

gives the nontrivial solutions

$$B = \begin{bmatrix} b & (b-1)^2 \\ -1 & 2-b \end{bmatrix},$$

where b is any complex number. Unlike in Ref. [6], where only a finite collection of solutions were found, we explore the solution set structure for some cases by finding an expression for infinitely many solutions. Our approach is based on several simple results and a format reduction where A in Eq. (1.1) is replaced by its Jordan canonical form, which simplifies the computation.

In Section 2, we present simple sufficient conditions under which a square matrix B is a solution to Eq. (1.1). In Section 3, we give the explicit expression of the solutions to Eq. (1.1) for several types of Jordan canonical form for A. A numerical example that contrasts our results here with those of Ref. [6] is discussed in Section 4, and our conclusions are in Section 5.

2. Sufficient Conditions for a Solution

Throughout this article, we assume that A is a fixed square $n \times n$ matrix. The set $\sigma(A)$ denoting all the eigenvalues of A is called the *spectrum* of A. An eigenvalue is said to be *semi-simple* if its algebraic multiplicity and geometric multiplicity are equal; and if both of these multiplicities equal one, the eigenvalue is called *simple*. An eigenvalue that is not semi-simple is called a *defective* eigenvalue. We now present several general sufficient conditions for solving Eq. (1.1), which are to be applied in the next section for structural analysis of the solutions for several particular matrices A.

Theorem 2.1. *If B is a matrix that satisfies $AB = BA = B^2$, then B is a solution of Eq.* (1.1).

Proof. From $AB = B^2$ we have $ABA = B^2A$. On the other hand, since $BA = AB$ we have $BAB = B^2A$, and hence B solves Eq. (1.1). □

An important special case of Theorem 2.1 is given in the following theorem.

Theorem 2.2. *If P is a projector that commutes with A, then the matrix $B = AP$ is a solution of Eq.* (1.1).

Proof. Since P commutes with A, $AB = A(AP) = (AP)A = BA$. Furthermore, since P is a projector, $B^2 = (AP)(AP) = A^2P^2 = A^2P = A(AP) = AB$, so the result follows from Theorem 2.1. □

As the following result shows, if A is nonsingular then the matrix B in Theorem 2.1 must be in the form described in Theorem 2.2.

Theorem 2.3. *Suppose A is nonsingular. Then any solution B obtained in Theorem* 2.1 *is of the form $B = AP$, where P is a projector that commutes with A.*

Proof. First notice that $AB = BA$ implies $A^{-1}B = BA^{-1}$. Let $P = A^{-1}B$. Then

$$AP = B = BA^{-1}A = A^{-1}BA = PA\,.$$

Furthermore, $AB = B^2$ implies $A^{-1}B^2 = B$. Hence

$$P^2 = A^{-1}BA^{-1}B = A^{-1}A^{-1}B^2 = A^{-1}B = P\,.$$

□

3. Solution Structures for Some Types of Yang-Baxter-like Equation

We next establish several explicit forms of infinitely many solutions to certain types of Eq. (1.1), as a direct consequence of the general results in the previous section. Firstly, it is shown that in order to solve Eq. (1.1) it is enough to solve the corresponding simpler case where A is replaced with its Jordan canonical form.

Lemma 3.1. *Let two $n \times n$ matrices A and B satisfy Eq.* (1.1). *Then for any $n \times n$ nonsingular matrix S, the matrices $A' = S^{-1}AS$ and $B' = S^{-1}BS$ satisfy*

$$A'B'A' = B'A'B'.$$

Furthermore, if B' satisfies this equation for a given $A' = S^{-1}AS$, then $B' = S^{-1}BS$ for some B such that $ABA = BAB$.

Proof. The result follows from the fact that Eq. (1.1) is satisfied if and only if

$$(S^{-1}AS)(S^{-1}BS)(S^{-1}AS) = (S^{-1}BS)(S^{-1}AS)(S^{-1}BS)\,.$$

□

Since every square matrix A is similar to its Jordan canonical form J, that is $A = SJS^{-1}$ for some nonsingular matrix S, it follows from Lemma 3.1 that a matrix K is a solution to Eq. (1.1) with $A = J$ if and only if matrix $B = SKS^{-1}$ is a solution. Thus without loss of generality, henceforth we assume that A is this Jordan canonical form — i.e. A is the block diagonal matrix J consisting of Jordan blocks.

We start with the simplest case to motivate ideas. Suppose $J = [\lambda] \in \mathbf{C}^{1\times 1}$ where $\lambda \neq 0$, and consider the corresponding one-dimensional form $\lambda x(\lambda - x) = 0$. Since $\lambda \neq 0$, $x = 0$ and $x = \lambda$ are the trivial solutions. Now let us consider a general nonsingular Jordan block

$$J = \begin{bmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix} \in \mathbf{C}^{l\times l}, \tag{3.1}$$

where $\lambda \neq 0$ and $l \geq 2$. Since J is nonsingular, by Theorem 2.3 we look for the solutions of the form $K = JP$, where P is a projector that commutes with J. A simple computation shows that all matrices P such that $JP = PJ$ are upper triangular Toeplitz — i.e. with all the entries along any diagonal line the same numbers. Thus letting P be such a matrix, if we denote its common main diagonal entry by μ then μ is the only eigenvalue of P. Since P is also a projector, it is diagonalisable, so there is a nonsingular matrix S such that $S^{-1}PS = \mu I$, and hence $P = \mu I$. The fact that P is a projector then ensures that μ is either 0 or 1, so either $P = 0$ or $P = I$ — hence either $K = 0$ or $K = J$, which are the trivial solutions. In summary, we have the following result.

Theorem 3.1. *Let J be given by* (3.1). *Then the solutions of the corresponding Yang-Baxter-like matrix equation obtained from Theorem* 2.1 *are only the trivial ones.*

However, if $J = [0] \in \mathbf{C}^{1\times 1}$ or $\lambda = 0$ in the matrix J defined in (3.1), we can find infinitely many solutions via Theorem 2.1 as follows.

Theorem 3.2. *Let $J = [0] \in \mathbf{C}^{1\times 1}$ or*

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 0 & 1 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \in \mathbf{C}^{l\times l}, \tag{3.2}$$

where $l \geq 2$. Then there are infinitely many solutions K to the corresponding Yang-Baxter-like matrix equation. Moreover, the explicit expression of such solutions is available.

Proof. If $J = [0]$, then all numbers k are solutions in this trivial case. When J is given by Eq. (3.2), we seek all solutions K that can be obtained from Theorem 2.1. The condition $JK = KJ$ for solutions K implies that K must be an upper triangular Toeplitz matrix of the form

$$K = \begin{bmatrix} k_1 & k_2 & \cdots & \cdots & k_{l-1} & k_l \\ 0 & k_1 & k_2 & \cdots & \cdots & k_{l-1} \\ 0 & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & k_2 \\ 0 & 0 & \cdots & \cdots & 0 & k_1 \end{bmatrix}.$$

The other requirement from Theorem 2.1 that $JK = K^2$ sets up the equation $(K - J)K = 0$ — i.e.

$$\begin{bmatrix} k_1 & k_2 - 1 & \cdots & \cdots & k_{l-1} & k_l \\ 0 & k_1 & k_2 - 1 & \cdots & \cdots & k_{l-1} \\ 0 & 0 & k_1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & k_1 & k_2 - 1 \\ 0 & 0 & \cdots & \cdots & 0 & k_1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & \cdots & \cdots & k_{l-1} & k_l \\ 0 & k_1 & k_2 & \cdots & \cdots & k_{l-1} \\ 0 & 0 & k_1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & k_1 & k_2 \\ 0 & 0 & \cdots & \cdots & 0 & k_1 \end{bmatrix} = 0\,.$$

By equating column by column on each side of this equality, it is not difficult to get the following formulas for K. If $l = 2$, then

$$K = \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix}; \tag{3.3}$$

if $l = 3$, then

$$K = \begin{bmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad K = \begin{bmatrix} 0 & 1 & k \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}; \tag{3.4}$$

and if $l \geq 4$ then

$$K = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & k \\ 0 & 0 & 0 & \ddots & & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ \vdots & & & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix} \tag{3.5}$$

or

$$K = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & k \\ 0 & 0 & 1 & \ddots & & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ \vdots & & & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \tag{3.6}$$

where k is an arbitrary number. This set of solutions of the Yang-Baxter-like matrix equation is a one-dimensional manifold. $\square$

Lemma 3.1 and Theorem 3.2 immediately imply the following result.

Corollary 3.1. *Suppose that* 0 *is a simple or defective eigenvalue of a matrix* A*. Then there are infinitely many solutions to Eq.* (1.1) *with explicit expressions.*

Proof. Since the Jordan canonical form of A is a block diagonal matrix with Jordan blocks along the diagonal, without loss of generality we may assume that the Jordan canonical form J of A is the first J in Theorem 3.1 if $\lambda = 0$ is simple or the second J in Theorem 3.1 if $\lambda = 0$ is defective. Then there is a nonsingular matrix S such that $A = SJS^{-1}$. Now let K be any matrix satisfying $JKJ = KJK$ obtained from Theorem 3.1. Then K is any scalar if $\lambda = 0$ is a simple eigenvalue, and K is given by one of the formulas (3.3)-(3.6) depending on the size of J, if $\lambda = 0$ is a defective eigenvalue. Consequently, from Lemma 3.1 we conclude that $B = SKS^{-1}$ solves Eq. (1.1). $\square$

Next, we consider the case when $\lambda \in \sigma(A)$ has at least two associated 1×1 Jordan blocks.

Theorem 3.3. *Suppose* $J = \mathrm{diag}(\lambda, \lambda, \cdots, \lambda) \in \mathbf{C}^{l\times l}$ *with* $l \geq 2$*. If* $\lambda \neq 0$*, then for any* $l \times l$ *projector* P *we have that* $K = \lambda P$ *solves the corresponding Yang-Baxter-like matrix equation. If* $\lambda = 0$*, then any* $l \times l$ *matrix* K *is a solution.*

Proof. The case $\lambda = 0$ is obvious. Suppose $\lambda \neq 0$. Since $J = \lambda I$, for any matrix K we have $JKJ = \lambda^2 K$ and $KJK = \lambda K^2$. Thus $JKJ = KJK$ if and only if $K^2 = \lambda K$, which is true if and only if $K = \lambda P$ with $P^2 = P$. $\square$

The proof of the next result is the same as that of Corollary 3.1.

Corollary 3.2. *Suppose that* $\lambda \in \sigma(A)$ *has at least two* 1×1 *Jordan blocks associated with it. Then there are infinitely many solutions to Eq.* (1.1) *with explicit expressions.*

Finally, we consider the case when $\lambda \in \sigma(A)$ is associated with at least two Jordan blocks, one of which is 1×1 and the other of which is $l \times l$ with $l \geq 2$.

Theorem 3.4. *Let J be an $(l+1)\times(l+1)$ Jordan canonical form with the structure*

$$J=\left[\begin{array}{c|cccccc} \lambda & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & \lambda & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 & \lambda \end{array}\right].$$

Then there are infinitely many solutions K to the corresponding Yang-Baxter-like matrix equation, which have explicit expressions.

Proof. First, assume that $\lambda \neq 0$. Then by Theorem 2.3 any solution described in Theorem 2.1 can be written as $K = JP$, where P is a projector that commutes with J. From the condition $JP = PJ$, we obtain that

$$P=\left[\begin{array}{c|cccccc} p & 0 & 0 & 0 & \cdots & a \\ \hline b & p_1 & p_2 & p_3 & \cdots & p_l \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & p_3 \\ \vdots & \vdots & & \ddots & \ddots & p_2 \\ 0 & 0 & \cdots & \cdots & 0 & p_1 \end{array}\right].$$

It is clear that p and p_1 are the only eigenvalues of P. Since P is a projector, its eigenvalues are only 0 and 1, so there are only four possibilities. If $p = p_1 = 0$, then $P = 0$, and hence $K = 0$, a trivial solution. If $p = p_1 = 1$, then $P = I$, and hence $K = J$, the other trivial solution. Suppose that $p = 1$ and $p_1 = 0$. Then from the condition $P^2 = P$, it follows that

$$P=\left[\begin{array}{c|cccccc} 1 & 0 & 0 & \cdots & 0 & a \\ \hline b & 0 & 0 & \cdots & 0 & ab \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{array}\right],$$

where a and b are arbitrary numbers. So

$$K=JP=\left[\begin{array}{c|cccccc} \lambda & 0 & 0 & \cdots & 0 & \lambda a \\ \hline \lambda b & 0 & 0 & \cdots & 0 & \lambda ab \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right] \tag{3.7}$$

is a solution to $JKJ = KJK$ for any a and b. Finally, let $p = 0$ and $p_1 = 1$. Then the condition $P^2 = P$ implies that

$$P = \left[\begin{array}{c|cccccc} 0 & 0 & 0 & \cdots & 0 & a \\ \hline b & 1 & 0 & \cdots & 0 & -ab \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right],$$

where a and b are arbitrary. Thus, if $l = 2$, then

$$K = JP = \left[\begin{array}{c|cc} 0 & 0 & \lambda a \\ \hline \lambda b & \lambda & 1 - \lambda ab \\ 0 & 0 & \lambda \end{array}\right]; \tag{3.8}$$

if $l = 3$, then

$$K = JP = \left[\begin{array}{c|ccc} 0 & 0 & 0 & \lambda a \\ \hline \lambda b & \lambda & 1 & -\lambda ab \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{array}\right]; \tag{3.9}$$

and if $l \geq 4$, then

$$K = JP = \left[\begin{array}{c|ccccccc} 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \lambda a \\ \hline \lambda b & \lambda & 1 & 0 & \cdots & 0 & 0 & -\lambda ab \\ 0 & 0 & \lambda & \ddots & \ddots & & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & & & & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \lambda \end{array}\right] \tag{3.10}$$

is a solution to the corresponding Yang-Baxter-like matrix equation for any a, b.

Now assume that $\lambda = 0$. In this case we use Theorem 2.1 directly. Since $JK = KJ$, we obtain

$$K = \left[\begin{array}{c|ccccc} k & 0 & 0 & 0 & \cdots & a \\ \hline b & k_1 & k_2 & k_3 & \cdots & k_l \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & k_3 \\ \vdots & \vdots & & \ddots & \ddots & k_2 \\ 0 & 0 & \cdots & \cdots & 0 & k_1 \end{array}\right].$$

The additional condition $JK = K^2$ means $(K - J)K = 0$. Equating each column of this matrix equation, we obtain that if $l = 2$, then

$$K = \left[\begin{array}{c|cc} 0 & 0 & a \\ \hline 0 & 0 & c \\ 0 & 0 & 0 \end{array}\right] \quad \text{or} \quad K = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline b & 0 & c \\ 0 & 0 & 0 \end{array}\right]; \tag{3.11}$$

if $l = 3$, then

$$K = \left[\begin{array}{c|ccc} 0 & 0 & 0 & a \\ \hline b & 0 & \frac{1\pm\sqrt{1-4ab}}{2} & c \\ 0 & 0 & 0 & \frac{1\pm\sqrt{1-4ab}}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]; \tag{3.12}$$

and if $l \geq 4$, then

$$K = \left[\begin{array}{c|ccccccc} 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & a \\ \hline b & 0 & 0 & 0 & \cdots & 0 & -ab & c \\ 0 & 0 & 0 & 0 & \ddots & & \ddots & -ab \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & 0 \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & & \ddots & \ddots & 0 \\ \vdots & \vdots & & & & & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{array}\right] \tag{3.13}$$

or

$$K = \left[\begin{array}{c|ccccccc} 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & a \\ \hline b & 0 & 1 & 0 & \cdots & 0 & -ab & c \\ 0 & 0 & 0 & 1 & \ddots & & \ddots & -ab \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & 0 \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & & \ddots & \ddots & 0 \\ \vdots & \vdots & & & & & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{array}\right] \tag{3.14}$$

is a solution to the corresponding Yang-Baxter-like matrix equation for any a, b, c. □

Corollary 3.3. *Suppose that $\lambda \in \sigma(A)$ is associated with at least two Jordan blocks in the Jordan canonical form of A, one of which is 1×1 and the other of which is $l \times l$ with $l \geq 2$. Then there are infinitely many solutions to Eq.* (1.1) *with explicit expressions.*

We end this section with the result that if $\lambda = 0 \in \sigma(A)$ then there are infinitely many solutions with explicit expressions.

Theorem 3.5. *If $\lambda = 0 \in \sigma(A)$, then there are infinitely many solutions to Eq.* (1.1) *which can be constructed explicitly.*

Proof. Suppose that $\lambda = 0 \in \sigma(A)$. The case when $\lambda = 0$ is simple or defective is covered in Corollary 3.1, and the case when $\lambda = 0$ is semisimple but not simple is covered in Theorem 3.3. So in all cases there are infinitely many solutions to Eq. (1.1). □

Remark 3.1. Indeed, any square matrix B such that $AB = 0$ is a solution of Eq. (1.1) when A is singular — and since there are infinitely many such matrices B, Theorem 3.5 follows immediately.

4. Numerical Example

Let us now consider a numerical example that contrasts our results with those in Ref. [6]. For any square matrix A, the index $\nu(\lambda)$ of its eigenvalue λ is the smallest nonnegative integer j such that $\ker((A-\lambda I)^{j+1}) = \ker((A-\lambda I)^j)$. From the main result in Ref. [6], the matrix $B = AP$ is a solution of Eq. (1.1) if P is the projector onto $\ker((A-\lambda I)^{\nu(\lambda)})$ along $\mathrm{ran}((A-\lambda I)^{\nu(\lambda)})$. .

Example 4.1. Consider

$$A = \begin{bmatrix} 3 & 2 & 2 \\ -2 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

One can verify that $A = SJS^{-1}$, where

$$J = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right] \quad \text{and} \quad S = \begin{bmatrix} 0 & 4 & 1 \\ 4 & -4 & 1 \\ -4 & 0 & 0 \end{bmatrix}.$$

Since 1 is the only eigenvalue of the matrix and $(A-I)^2 = 0$, it follows that $P = I$ — and hence the solution B obtained from the spectral solution theorem of Ref. [6] is the trivial solution A. However, we can now construct infinitely many solutions to Eq. (1.1) according to the results here. From Theorem 3.4 and Eqs. (3.7) and (3.8) ($\lambda = 1$ and $l = 2$), we have two two-parameter families of solutions $B = SKS^{-1}$ to Eq. (1.1) — viz.

$$K = \left[\begin{array}{c|cc} 1 & 0 & a \\ \hline b & 0 & ab \\ 0 & 0 & 0 \end{array}\right] \quad \text{and} \quad K = \left[\begin{array}{c|cc} 0 & 0 & a \\ \hline b & 1 & 1-ab \\ 0 & 0 & 1 \end{array}\right]$$

respectively, where a and b are arbitrary numbers. Expanding SKS^{-1} and rewriting $2a$ as

a since a is arbitrary, we have the two families of solutions

$$B_1(a,b) = \begin{bmatrix} ab & ab & (a-1)b \\ a(1-b) & a(1-b) & (a-1)(1-b) \\ -a & -a & -(a-1) \end{bmatrix}$$
$$= \begin{bmatrix} b & 0 & 0 \\ 0 & 1-b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a-1 \end{bmatrix}$$

and

$$B_2(a,b) = \begin{bmatrix} 3-ab & 2-ab & 2-(a+1)b \\ a(1+b)-2 & a(1+b)-1 & -2+(1+a)(1+b) \\ -a & -a & -a \end{bmatrix},$$

where a and b are arbitrary numbers. It is clear that $B_2(a,b) = A - B_1(-a,-b)$ in the above example, and In general one can easily prove the following result:

Theorem 4.1. *Under the condition of Theorem* 2.1*,* $A - B$ *is also a solution of Eq.* (1.1) *and belongs to the solution set obtained from Theorem* 2.1.

5. Conclusions

Using simple criteria for some solutions of the Yang-Baxter-like matrix equation (1.1) and assuming several special structures of the Jordan canonical form for $\lambda \in \sigma(A)$, we obtained infinitely many explicit solutions that commute with A and completely characterised them. It will be an interesting task to explore the solution structure for other types of Jordan structures corresponding to $\lambda \in \sigma(A)$ — in particular, solutions that do not commute with the given matrix A.

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